11. Fatigue of Metallic Materials

11.1 Introduction (What we're gonna do here)

Metal fatigue of mechanical parts and structural elements refers to mechanical failure by fracture due to the formation and growth of cracks that are formed under cyclic (dynamic) loads, see Figure 1. But you already knew that, did you? This should have been outlined already in multiple courses. Anyway, this chapter of the HSRW-SoM lecture notes goes through it all again. This time, with the objective of assembling all the previous topics into something understandable that can be applied to actually do mechanical design against fatigue. You have probably heard about the Schmidt-diagram, fracture toughness of materials and Charpy-V drop hammer testing, the German FKM-guideline and possibly even fracture mechanics. Your mechanics professor, however, isn't particular good at any of these items, and is happy that they all are considered elsewhere. So this chapter is written in order to avoid the above-mentioned topics and outline a design approach after briefly showing how to obtain reference stress measures and how to get principal stresses using eigenvalue-problems (because these are everywhere).

The crucial aspect in the current approach is to account for notch stresses and the effect of a non-zero mean stress component in our designs. This implies that we when we're done, we'll know how to handle a load which is not fully reversed along with stress concentrations caused by sudden changes in geometry. The effect of a non-zero mean stress is visualized in Figure 2 considering the number of load cycles applied until failure occurs in a test specimen. Compressive mean stresses can be observed to be less critical than tensile mean stresses. This also makes sense qualitatively speaking, since compressive stresses will tend to close surface cracks, while tensile stresses will open them and lead to progressive formation of cracks. This is why crack formation, from a manufacturing perspective, can be prevented by hammering, shot peening, etc., since these processes introduce residual compressive stresses in the treated surface.[1]

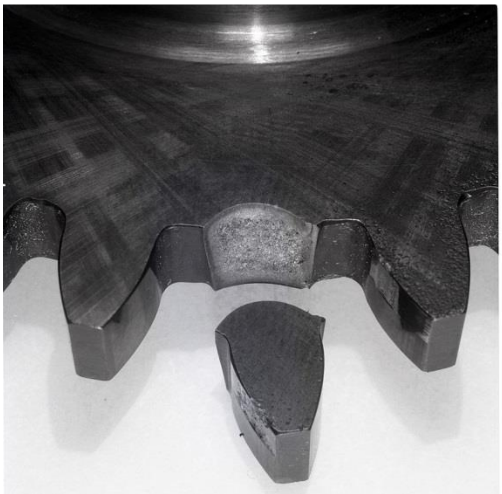


Figure 1 Fatigue failure in the tooth of a helical gear, Photo from Wikimedia

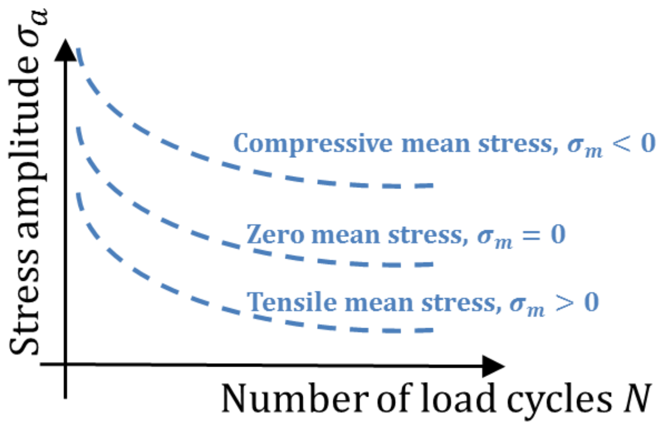


Figure 2 Number of load cycle until failure for various cyclic stresses with varying mean stress

Fatigue failure is admittedly among the trickiest aspects of mechanical design and often the one that is most confusing in the intro engineering classes. This is actually not particular surprising, if one thinks about the complexity of the considered phenomena. When considering crack growth, one approach is to analyze the stress states around cracks formed by various fracture mechanisms

[1] An example of how these processes effect the lifetime of welded parts is available here

using differential equations in order to obtain mathematical models, see Figure 3. This has led to the discipline called fracture mechanics and produces beautiful and complex mathematical models of fatigue as failure mechanism. Another approach to the study of crack growth is based on not thinking too much about fracture mechanisms and differential equations and conduct a high (like very high) number of experiments by which the number of load cycles until failure occurs simply is counted for various cyclic loads, see Figure 4. While the first approach is in the regime of theoretical applied mechanics, the second approach is followed by material scientists doing experimental research; and neither are particularly practical to apply when designing stuff. So it's kind of up to us to come with something practical here. Luckily, there are a high number of textbooks on the topic, for example [1]-[3].

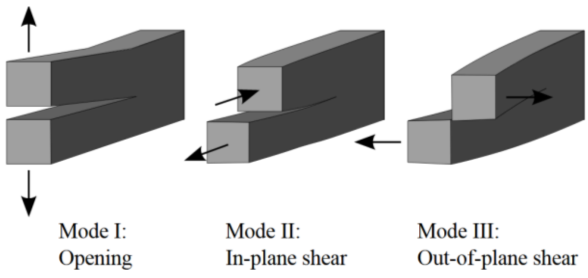


Figure 3 Crack mode commonly studied in fracture mechanics, Photo from Wikimedia

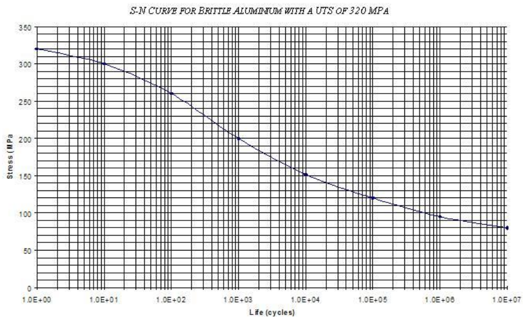


Figure 4 Experimental results for the life time of fatigue test samples (S-N curve), Photo from Wikimedia

Uncertainty in loads, which a mechanical part or element will experience – in particular when considering natural loads like wind, waves, current etc., is a particular challenge, which will not be considered in this chapter (there'll hopefully be a chapter about this at some point in the future). Furthermore, we're going to assume that the fatigue lifetime of a mechanical part or structural element is independent of the strain velocity (how fast loads are applied), the actual fracture mechanism (crack mode) and in which sequence varying loads are applied. Obviously, all three of the statements are lies. As a consequence, it is amazing how well the strategies outlined in the following actually work when applied to actual real life designs. The current scope is limited to linear elastic stress ranges in metallic materials. Or else, we'd never make our way through this topic.

11.2 Principal- and reference stress

11.2.1 Reference stress

The first thing we'll briefly recap is the basic question of when a material subjected to both normal and shear stress will fail. This question has turned out to not really have a simple answer, and as a consequence, a high number of failure criteria have been developed for specific types of materials[2]. However, the most widely applied failure criteria for ductile metallic materials is probably von Mises reference stress, which for a 3D state of mixed stresses is given by

[2] Though it still academically speaking is considered bad style to cite Wikipedia, the material failure article actually provides a rather good list of failure criteria for various materials – that sort of should give an idea of how complex the matter is, and the formulation of those have indeed led to arguments in the past in the academic communities having almost the nature of ideological or religious wars. So caution is advised in discussions about failure criteria.

$$\sigma_v = \sqrt{\frac{(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)}{2}} \qquad 1.$$
$$= \sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}}$$

in which the second line is rewritten into principal stresses. We recall, that these occur when the stress state is described using a coordinate system chosen in such a way that the shear stresses vanish leaving us only with the numerically largest possible normal stresses. Following this approach, a single stress value σ_v can be defined and compared to an allowable stress – often related to the yield limit determined by a tensile test.

11.2.2 Principal stress

A feature, which we haven't really considered yet, is the calculation of principal stresses on the basis of proper maths. This far, we have solely considered this problem using the stress transformation equations, often visualized by Mohr's circle [7]. Basically, we may rotate any stress state by application of transformation matrices

$$[\sigma]_{rot}=[T][\sigma][T]' \qquad 2.$$

in which the transformation matrix and the stress tensor is given by

$$[T] = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \quad [\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \qquad 3.$$

Our objective is to determine the rotation that causes the non-diagonal shear entries of the stress tensor to vanish. It can be shown mathematically, that this corresponds to calculating the eigenvalues of the stress tensor arising from the characteristic polynomial in the form

$$|[\sigma] - \lambda[I]| = 0 \qquad 4.$$

The principal directions can, in an equivalent way, be determined as the eigenvectors. This method for calculation of principal stresses is often quite practical and fairly easy to implement, also for 3D stress states, in software for scientific programming.

11.2.3 Calculated example 11A

A 2D stress state with $\sigma_x = 60\ \mathrm{N/mm^2}$, $\sigma_y = 40\ \mathrm{N/mm^2}$ and $\tau_{xy} = 30\ \mathrm{N/mm^2}$ will be considered, see Figure 5. Our objective is to determine the principal stresses and directions using a) Mohr's circle and b) the eigenvalues of the stress tensor, see equation 4. Furthermore, we'll calculate the von Mises reference stress.

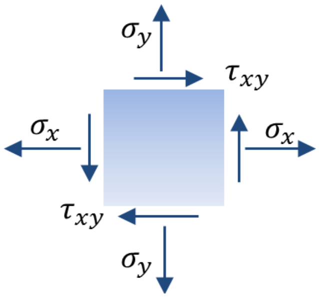


Figure 5 The considered state of plane stress

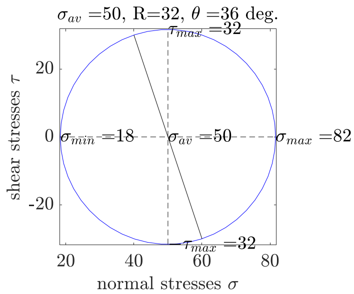


Figure 6 Mohr's circle

a) The average normal stress and the radius of Mohr's circle is given by

$$\sigma_{av} = \frac{\sigma_x+\sigma_y}{2} = \frac{60+40}{2}\frac{N}{mm^2} = 50\ \mathrm{N/mm^2} \qquad R = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2} = \sqrt{\left(\frac{60-40}{2}\right)^2 + (30)^2}\frac{N}{mm^2} = 32\ \mathrm{N/mm^2}$$

The principal stresses can now be calculated

$$\sigma_1 = \sigma_{av} + R = 82\ \mathrm{N/mm^2} \qquad \sigma_2 = \sigma_{av} - R = 18\ \mathrm{N/mm^2}$$

Reviewing Mohr's circle, see Figure 6, the required counter clockwise angle of rotation to obtain the principal direction

$$\tan(2\theta) = \frac{2\tau_{xy}}{\sigma_x-\sigma_y} = \frac{2\cdot 30}{60-40} \rightarrow \theta = 36\ \mathrm{deg}$$

In which we recall that a physical rotation of θ, corresponds to a rotation of 2θ in Mohr's circle.

If we want to figure out if this small piece of metal fails, we may calculate the von Mises reference stress. If we base this on the original stress state before rotation in 2D, we have

$$\sigma_v = \sqrt{{\sigma_x}^2 + {\sigma_y}^2 - \sigma_x\sigma_y + 3{\tau_{xy}}^2} = \sqrt{60^2 + 40^2 - 60\cdot 40 + 3\cdot 30^2}\frac{\mathrm{N}}{\mathrm{mm^2}} = 74.2\frac{\mathrm{N}}{\mathrm{mm^2}}$$

We can, just to check the validity of the calculation, check this in principal coordinates

$$\sigma_v = \sqrt{{\sigma_1}^2 + {\sigma_2}^2 - \sigma_1\sigma_2} = \sqrt{82^2 + 18^2 - 82\cdot 18}\frac{\mathrm{N}}{\mathrm{mm^2}} = 74.2\frac{\mathrm{N}}{\mathrm{mm^2}}$$

So this fits beautifully. We have here obtained a single stress value we could compare with an allowable normal stress – in lack to something better we'd often use the yield stress, to calculate a safety factor.

b) Having now solved this problem using the classical approach, let's try to do this using the eigenvalues

$$|[\sigma] - \lambda[I]| = 0 \rightarrow \left|\begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right| = 0 \rightarrow \left|\begin{bmatrix}\sigma_x-\lambda & \tau_{xy}\\ \tau_{xy} & \sigma_y-\lambda\end{bmatrix}\right| = 0$$

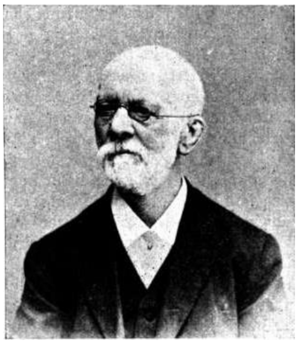
$$\rightarrow (\sigma_x - \lambda)(\sigma_y - \lambda) - \left(\tau_{xy}\right)^2 = \lambda^2 - \left(\sigma_x + \sigma_y\right)\lambda + \sigma_x\sigma_y - \left(\tau_{xy}\right)^2 = 0$$

$$\sigma_{1,2} = \lambda_{1,2} = \frac{(\sigma_x+\sigma_y)\pm\sqrt{\left(\sigma_x+\sigma_y\right)^2-4\left(\sigma_x\sigma_y-\left(\tau_{xy}\right)^2\right)}}{2} = \begin{cases}82\\ 18\end{cases}\frac{N}{mm^2}$$

We can now redo the entire problem using Matlab.

```
sigmax=60; sigmay=40; tau=30;    %Input plane state of stress
A=[sigmax,tau;tau,sigmay];        %Define stress tensor
[D,lambda]=eigs(A);               %Calculate principal stresses
sigma1=(lambda(1,1))
sigma2=(lambda(2,2))
v1=[D(1,1),D(2,1)];               %Calculate principal directions
tht1=acosd(dot(-[1,0],v1)/norm(v1))
v2=[D(1,2),D(2,2)];
tht2=acosd(dot(-[1,0],v2)/norm(v2))
```

11.3 Metallic materials: High cycle fatigue life time of un-notched specimens



August Wöhler (1819-1914):
German railway engineer and a great guy, who originally got the idea of performing lots and lots of tests with specimens of standardized size subjected to cycle loads to measure when these fail due to fatigue. Afterwards the measured point cloud turned out to look less messy and scary if plotted in a double logarithmic coordinate system, which ever since has been a standard trick in engineering to make the scatter seem less overwhelming.
S-N curves are often called Wöhler curve.
(Photo from Wikimedia, public domain)

11.3.1 The S-N curve for standard test specimens

If standardized tensile test specimens are subjected to various fully reversed cyclic loads (meaning having zero mean) until failure, we can generate a point cloud showing where each sample failed. When plotted in a double logarithmic coordinate system, most steel types would produce results as shown in Figure 7. Aluminum exhibit a similar behavior, except that the flattening effect referred to as the knee is not present, but the so-called Basquin line just continues straight to hell. This type of curve is called a S-N or Wöhler curve and is the primary metric for material scientists to measure how many load cycles a certain material can sustain before failing from fatigue due to crack formation. For steel types, the so-called knee clearly reveals that a load level exists for which cracks will never grow and theoretically speaking the considered test specimen will have infinite fatigue lifetime. This particular stress level is called the unmodified endurance limit when measured for standardized test specimens and occurs when about 10^6 load cycles have been applied.

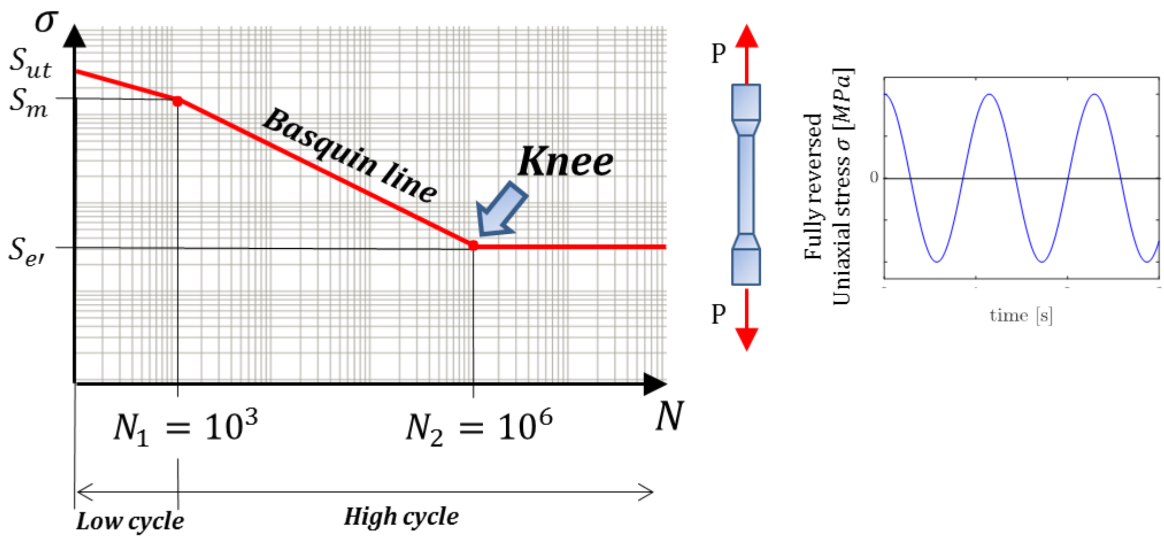


Figure 7 Steel S-N curve based on measured fatigue failure of standardized specimens

We distinguish between *low cycle fatigue*, when failure occurs before 10^3, and high cycle fatigue governed by the so-called Basquin line, for which failure occurs after 10^3 load cycles have been

applied. If cyclic loads causing a test specimen to fail within the first 1000 cycles of a test, a significant level of plasticity will occur in the test specimen. This isn't particularly practical when designing on the basis of our current stress paradigm, since we are kind-of hung up on linear elastic stresses. Therefore, low-cycle fatigue analysis is usually performed based on strain calculations, in which the strain components are divided into an elastic and a plastic component. We're not going to go there. We will limit ourselves to the high-cycle fatigue regime, since this eventually is what we mainly need for mechanical design.
In the high-cycle fatigue region, the Basquin line is given by the equation

$$\log(\sigma) = \log(a) + b\log(N) \qquad 5.$$

in which the constants a and b are determined based on tensile tests. This is an accurate and scientifically fulfilling, but also extremely time consuming process. Therefore, often for mechanical design, we would have to count on being able to get these values based on material specifications, since only very high-level engineering applications usually justifies the generation of specific S-N curves by experimental means for a given project. Luckily, there's a commonly applied hack for steel specimens. Statistically speaking, the material strength shows a quite decent correlation with the ultimate (breaking) strength of the material

$$S_m = \begin{cases} 0.90S_{ut} & (bending) \\ 0.75S_{ut} & (axial\ loads) \end{cases} \qquad 6.$$

The unmodified endurance limit exhibit a similar good correlation

$$S_{e'} = \begin{cases} 0.50S_{ut} & S_{ut} < 1400\ N/mm^2 \\ 700\ N/mm^2 & S_{ut} > 1400\ N/mm^2 \end{cases} \qquad 7.$$

The hacks applied in equation 6 and 7 do not provide a particularly high accuracy compared to actual measurements or data from material specifications, but are often very useful in design in lack of something better.

11.3.2 The modified S-N curve

We can now, by various means, obtain a S-N curve for high-cycle fatigue calculations of tensile test specimens. The next question to ask is whether this curve is generally valid for all mechanical elements and structural members made of the same material. It would have been really great if that was the case, but we are not that lucky. It turns out that a number of factors influence the endurance limit. We will therefore define the modified endurance limit as the parameter valid for a particular part by

$$S_e = C_{load}C_{size}C_{surf}C_{temp}C_{rel}S_{e'} \qquad 8.$$

The five necessary factors required will be explained in the following.

The load factor It turns out that various loading conditions affect the endurance limit differently. We have

$$C_{load} = \begin{cases} 1 & \text{(bending)} \\ 0.7 & \text{(axial loads)}^{*1} \\ 0.577 & \text{(torsion)}^{*2} \\ 1 & \text{(combined loads – use von Mises)} \end{cases} \qquad 9.$$

*1) Some sources actually are to set the load factor to 0.8 or even 0.85.
*2) If von-Mises reference is applied for stress calculation in torsion, $C_{load} = 1$ can be applied.

The size factor (by far the trickiest one) accounting for the fact that mechanical and structural elements may be significantly larger than the 8 mm standardized test samples usually applied for cyclic tensile and rotatory bending fatigue tests. I.e. the probability of material faults increases with the size of the considered part, and this reduces the fatigue endurance strength.

Furthermore, in a small specimen in bending, the stress will decay faster when moving from the outer surface into the material than for a larger specimen.

For rotary bending tests of solid and hollow cylindrical steel parts, it has turned out that the size factor can be calculated by the simple rule obtained by fitting measurements

$$C_{size} = \begin{cases} 1 & d \leq 8mm \text{ – or part in tension} \\ 1.189 \cdot d^{-0.097} & 8mm < d \leq 250mm \\ 0.6 & d > 250mm \end{cases} \qquad 10.$$

That wasn't too hard. Anyway, this only goes for steel cylinders in rotary bending, though these factors are also commonly used in torsion. Anyway, for axial loads, we may use $C_{size} = 1$ (pure tension). But the question of how to handle non-rotary bending still remains open. Here comes a really wicked idea (originally proposed by a guy called Kuguel – I admit I've never read his paper). Fatigue crack growth is governed by the cross-sectional regions containing the largest stresses. Let's equate the area of a cylindrical part and the area of any other part considered over which stress levels equal 95% or more of the max. stress level occurs, and use that area as the basis for a reference diameter, we'll simply toss it directly into equation 10. This particular reference area is given by

$$A_{0.95} = \frac{\pi}{4}(d^2 - (0.95d)^2) = 0.0766d^2 \qquad 11.$$

We can now calculate an equivalent diameter to apply in equation 10

$$d_{equiv} = \sqrt{\frac{A_{0.95}}{0.0766}} \quad \text{(rotating round)} \qquad 12.$$

Using this particular (and quite wicked) way of thinking, we obtain the following equivalent diameters for the most common cross sections

$$d_{equiv} = 0.37d \quad \text{(non-rotating round)} \qquad 13.$$

$$d_{equiv} = 0.808\sqrt{bh} \quad \text{(non-rotating rectangular)} \qquad 14.$$

$$d_{equiv} = 0.10bt_f \quad \text{(non-rotating I-beam)} \qquad 15.$$

The surface factor is added to account for the fact that surfaces with a rough finish perform worse in fatigue than surfaces with a fine finish – and how coarse or fine a surface is depends on the process used to manufacture it. We finally have found something here that's easy to grasp. A statistical fit provides the expression

$$C_{surf} = a(S_{ut})^b \qquad 16.$$

In which a and b are process dependent constants that can be found in Table 1.

Surface finish	**a**	**B**
Ground	1.58	-0.085
Machined or cold-drawn	4.51	-0.265
Hot-rolled	57.7	-0.718
As-forged	272	-0.995

Table 1

Reliability (%)	C_{rel}
50	1.000
90	0.897
99	0.814
99.9	0.753
99.99	0.702
99.999	0.659

Table 2

The temperature factor accounts for the fact that high temperatures may reduce the endurance limit. In the current context, we'll simply notice that we may set $C_{temp} = 1$ for temperatures $T < 450°C$

The reliability factor may be added to account for the fact that the fatigue material parameters are obtained with standard statistical means, meaning that there's a 50% possibility that the

actual parameters for a given test sample are worse. If a high reliability is required, this factor can be added. Commonly used values reproduced from [3] are given in Table 2.
Environmental effects, like reduction of endurance limits due to presence of chloride or acid, is a bit trickier to account for. In some sources, this is done by adding an additional factor C_{envi} to equation 8. However, this is indeed a very inaccurate way to account for these effects and in general, it is recommended to obtain and apply specific parameters obtained by testing.

11.4 Fatigue life time of notched mechanical parts with non-zero mean stress

All the hard work we just went through in the previous section was all about figuring out what the material strength and the endurance limit of some metallic material is. We haven't even started considering how to calculate the actual stresses in a given mechanical part or structural member. The main thing we'll have to work out here is how to handle the effect of rapid changes in geometry, which we call *notches*, see Figure 8 and Figure 9. For static loads we already know, that there's a thing called the stress concentration (or amplification) factor, which we multiply with to get the stress induced by the notch. However, for cyclic loading of ductile materials, notches turn out to be a bit less harmful than for static loads.

11.4.1 Recap: the static stress concentration factor for notched specimens

We know from the basic class on strength of materials, that the stress concentration factor K_t for static loading conditions is dependent only on geometric parameters. Having derived this, or looked it up in tables, usually [8] – see Appendix A for a few basic examples. The notch stress is then given by

$$\sigma_{notch} = K_t \sigma_{nom} \qquad 17.$$

The nominal stress is calculated as usual based on the internal forces.

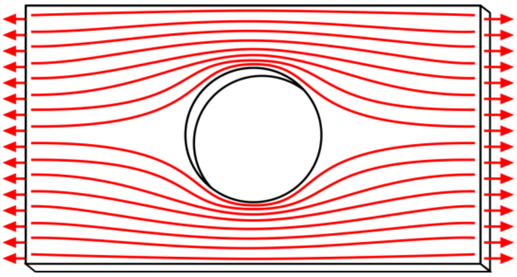


Figure 8 Load lines generating notch stress around hole in plate, from Wikimedia, CC-licensed

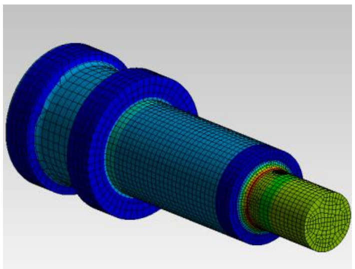


Figure 9 Notch stress due to abrupt change in diameter

11.4.2 The fatigue stress concentration factor

It turns out that ductile materials under cyclic loads are slightly more forgiving than when subjected to static loads. As a consequence, we may reduce the stress concentration factor leading to a second definition of the term known as the fatigue stress concentration factor K_f

$$K_f = 1 + q(K_t - 1) \qquad 18.$$

in which q denotes the notch sensitivity – a material constant dependent on the ductility of the material. The notch sensitivity can be obtained using the following expression suggested by a guy called Peterson

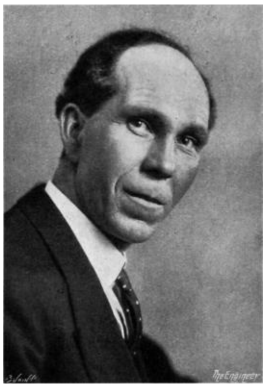
$$q = \frac{1}{1 + \frac{a}{r}} \qquad 19.$$

In which a is the characteristic length of a given material. For steel, this can be correlated (again!) with the ultimate strength using the following expression

$$a = 0.0254 \left(\frac{2070}{S_{ut}}\right)^{1.8} \quad 20.$$

with S_{ut} in MPa and a in mm. For aluminum alloys, a is often estimated to be 0.635 mm [2]. It is noted that these relations are purely empirical and based on fully reversed loading.

11.4.3 Effect of non-zero mean stress: the Haigh diagram



Bernard Haigh (1884-1941)
A Scottish professor at the British Naval college, who did a fantastic job not only describing the effects of non-zero mean stress of cyclic loads on the endurance limit ... but also contrary to many other people communicated this in a way understandable to the common engineer – see the Haigh diagram below.

(Photo from Wikimedia, GFDL)

There's a problem we haven't considered yet, namely that mean stress is not necessarily zero. A tensile mean stress will reduce the endurance limit (since it causes surface cracks to open) while it's safe to assume that a compressive mean stress will leave the endurance limit unchanged. In the classes on mechanical design, most students have learned to account for this effect using the so-called Schmidt diagram. However, the information contained in this particular visualization can be presented in a different way: the Haigh diagram, see Figure 10. In this type of visualization, the mean stress is being plotted against the amplitude stress.

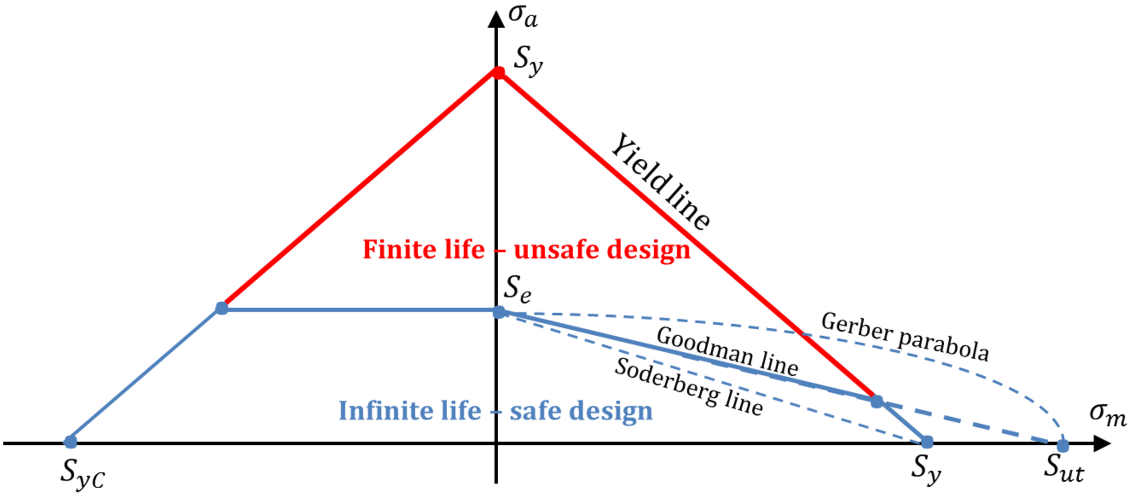


Figure 10 Haigh diagram

Now, we just have to work out how to define the line limiting a safe design in the tensile region, meaning the limit for which a mechanical part will have infinite lifetime. This is essentially about connecting the modified endurance limit for zero mean stress with an appropriate value on the horizontal axis (and was for many years the easiest way to immortalize your name in mechanical engineering – just draw another line in the tension region)

$$\text{Goodman line:} \quad \sigma_a = S_e\left(1 - \frac{\sigma_m}{S_{ut}}\right) \qquad 21.$$

The most conservative estimate is based on the yield stress and is given by

$$\text{Soderberg line:} \quad \sigma_a = S_e\left(1 - \frac{\sigma_m}{S_y}\right) \qquad 22.$$

Finally, a parabolic fit actually is what fits the test data best, though constituting the least conservative option

$$\text{Gerber parabola:} \quad \sigma_a = S_e\left(1 - \frac{\sigma_m^2}{S_{ut}^2}\right) \qquad 23.$$

11.4.4 Calculated example 11B

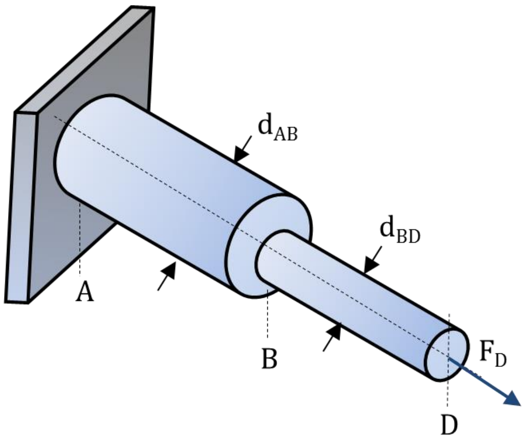


Figure 11

A compound cylinder has diameters $d_{AB} = 120\,mm$ and $d_{BD} = 80\,mm$ with a transition between the small and large diameter having a fillet with radius $r = 4\,mm$. The cylinder is made of machined steel with yield strength $S_y = 355\,N/mm^2$ and ultimate tensile strength $S_{ut} = 600\,N/mm^2$.

If a fully reversed cyclic axial load $F_D = 350\,kN$ is applied, determine:

a) The safety factor against static yielding
b) The safety factor against fatigue failure (assuming infinite lifetime is required)

If a constant static load $F_D^{(m)} = 300\,kN$ is applied in addition to the cyclic load mentioned above, determine:

c) The safety factor against fatigue failure (assuming infinite lifetime is required)

a) It is noted, that the internal force is constant, i.e. $F_{AB} = F_{BD} = F_D$. The nominal amplitude stresses are given by

$$\sigma_{AB}^{(a)} = \frac{F_{AB}}{\frac{\pi}{4}d_{AB}^2} = \frac{350\cdot 10^3\ \mathrm{N}}{\frac{\pi}{4}(120\ \mathrm{mm})^2} = 30.9\,\frac{\mathrm{N}}{\mathrm{mm}^2} \quad \sigma_{BD}^{(a)} = \frac{F_{BD}}{\frac{\pi}{4}d_{AB}^2} = \frac{350\cdot 10^3\ \mathrm{N}}{\frac{\pi}{4}(80\ \mathrm{mm})^2} = 69.6\,\frac{\mathrm{N}}{\mathrm{mm}^2}$$

The highest stress in the mechanical part will occur in the notch due to the transition between the small and the large diameter segment. The static stress concentration factor can be determined on the basis of the diagrams in Appendix A (Case 1).

$$\left.\begin{array}{l} \frac{D}{d} = \frac{d_{AB}}{d_{BD}} = \frac{120}{80} = 1.5 \\ \frac{r}{d} = \frac{r}{d_{BD}} = \frac{4}{80} = 0.05 \end{array}\right\} \rightarrow K_t \approx 2.25$$

The notch stress and corresponding safety factor against failure by yielding can now be calculated

$$\sigma_{B,static}^{(notch)} = K_t\sigma_{BD}^{(a)} = 2.25\cdot 69.6\,\frac{\mathrm{N}}{\mathrm{mm}^2} = 156.7\,\frac{\mathrm{N}}{\mathrm{mm}^2} \rightarrow SF_{yield} = \frac{S_y}{\sigma_B^{(notch)}} = \frac{355}{156.7} = 2.27$$

b)

Endurance strength: Having no specified information about the S-N curve of the material, the unmodified endurance strength is approximated quite accurately by equation 7 (for $S_{ut} < 1400\ \mathrm{N/mm^2}$):

$$S_{e'} = 0.50 S_{ut} = 0.5\cdot 600\,\frac{\mathrm{N}}{\mathrm{mm}^2} = 300\,\frac{\mathrm{N}}{\mathrm{mm}^2}$$

That would constitute the endurance strength of a tensile test sample. In order to modify this value to be representative for the actual mechanical part, we calculate the required factors:

$C_{load} = 0.7$ and $C_{size} = 1.0$ for axial loads.

For machined steel, we find $a = 4.51$ and $b = -0.265$ in Table 1, and obtain the surface factor using equation 16 to

$C_{surf} = a(S_{ut})^b = 4.51(600\ \mathrm{N/mm^2})^{-0.265} = 0.83$.

For service at ambient temperature and normal reliability, $C_{temp} = C_{rel} = 1$. The modified endurance strength can now be calculated applying equation 8:

$$S_e = C_{load}C_{size}C_{surf}C_{temp}C_{rel}S_{e'} = 0.7\cdot 1.0\cdot 0.83\cdot 1\cdot 1\cdot 300\,\frac{\mathrm{N}}{\mathrm{mm}^2} = 174\,\frac{\mathrm{N}}{\mathrm{mm}^2}$$

Fatigue notch stress: We observed in section 11.4.2, that when it comes to notch stresses, materials might be more forgiving when subjected to cyclic loads than when loaded statically. This is however dependent on how ductile, or brittle a material is. For steel, the notch length will be determined using equation 20 enabling us to calculate the notch sensitivity factor with equation 19:

$$a = 0.0254\left(\frac{2070}{S_u}\right)^{1.8} = 0.0254\left(\frac{2070}{600}\right)^{1.8} = 0.24\text{ mm} \qquad q = \frac{1}{1+\frac{a}{r}} = \frac{1}{1+\frac{0.24}{5}} = 0.94$$

The fatigue stress concentration factor is now given by

$$K_f = 1 + q(K_t - 1) = 1 + 0.94(2.25 - 1) = 2.18$$

The fatigue notch stress amplitude is

$$\sigma_B^{(a)} = K_f\sigma_{BD}^{(a)} = 2.18 \cdot 69.6\frac{\text{N}}{\text{mm}^2} = 151.8\frac{\text{N}}{\text{mm}^2}$$

The mean stress $\sigma_B^{(m)} = 0$, so the safety against fatigue failure can be calculated directly as

$$SF_{fatigue,fully\ reversed} = \frac{S_e}{\sigma_B^{(a)}} = \frac{174}{151.8} = 1.15$$

Strictly speaking, this safety is sufficient, but not particularly impressive. Most guidelines for steel design would require a higher safety factor.

c) Adding a constant stress component does not alter the amplitude stress (or cyclic component). It is still $\sigma_B^{(a)} = 151.8\text{ N/mm}^2$. However, we now have a non-zero mean stress. This may be calculated along with the corresponding notch stress

$$\sigma_{BD}^{(m)} = \frac{F_{BD}^{(m)}}{\frac{\pi}{4}d_{AB}^2} = \frac{300\cdot 10^3\text{ N}}{\frac{\pi}{4}(80\text{ mm})^2} = 59.7\frac{\text{N}}{\text{mm}^2} \qquad \sigma_B^{(m)} = K_f\sigma_{BD}^{(m)} = 2.18 \cdot 59.7\frac{\text{N}}{\text{mm}^2} = 130.1\frac{\text{N}}{\text{mm}^2}$$

The deal is that we will use the lines in the Haigh diagram to actually figure out if the design is safe. We'll start out using the **Goodman-line**, since this is the most commonly applied criteria. Applying equation 21 directly, on the basis of the calculated mean stress, we may determine the allowable stress amplitude

$$\sigma_{allow}^{(a)} = S_e\left(1 - \frac{\sigma_m}{S_{ut}}\right) = 174\frac{\text{N}}{\text{mm}^2}\left(1 - \frac{130.1}{600}\right) = 136.1\frac{\text{N}}{\text{mm}^2}$$

The safety factor is now given by

$$SF_{fatigue,Goodman} = \frac{\sigma_{allow}^{(a)}}{\sigma_B^{(a)}} = \frac{136.1}{151.8} = 0.90$$

With a safety below 1.0, the design is not safe meaning that the part will not have infinite lifetime. Now, let's presume that this part isn't critical from a safety perspective (in the sense that failure poses no potential for anyone getting hurt) and that the cost and consequence of failure is limited. We may try to calculate the safety factor applying the **Gerber parabola** in equation 23, which, after all, is the best fit to measured data:

$$\sigma_{allow}^{(a)} = S_e\left(1 - \frac{\sigma_m^2}{S_{ut}^2}\right) = 174\frac{\text{N}}{\text{mm}^2}\left(1 - \frac{130.1^2}{600^2}\right) = 165.7\frac{\text{N}}{\text{mm}^2}$$

This gives us a safety factor of

$$SF_{fatigue,Gerber} = \frac{\sigma_{allow}^{(a)}}{\sigma_B^{(a)}} = \frac{165.7}{151.8} = 1.09$$

So this might work, but the design is on the limit and recalling, that we have not decreased the reliability factor, the mechanical part should be resized to ensure that it can carry the prescribed loads.

11.5 Cycle counting and load spectra

[To be updated] … This is going to be about Rainflow counting and Palmgren-Miner's rule for linearly accumulated damage. That'll be next time.

11.6 Introduction to plasticity

[To be updated] … This will be about perfectly elastic-plastic material models and Ramberg-Osgood's equation for hysteresis and modeling of strain-hardening. That'll be next year.

References

[1] Jaap Schijve, Fatigue of Structures and Materials, Springer 2008

[2] RI Stephens et. Al, Metal Fatigue in Engineering, John Wiley & Sons 2001

[3] RL Norton, Machine Design – an integrated approach, 2nd ed. Prentice Hal 2000

[4] MM Pedersen, Introduction to Metal Fatigue – Concepts and Engineering Approaches, *Technical Report Mechanical Engineering*, 5(11), Aarhus University 2018

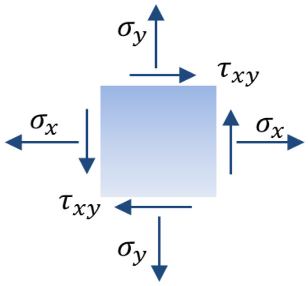
[5] N.H. Østergaard, Introduction to matrix methods in structural mechanics, rev. 01

[6] N.H. Østergaard, Lecture notes on Strength of Materials, ch. 3 – shafts in torsion

[7] N.H. Østergaard, Lecture notes on Strength of Materials, ch. 7 – transformation of plane stress

[8] W. Young and R. Budynas, Roark's Formulaes for stress and strain, 7th ed. McGraw-Hill, 2002

Problems



Problem 1

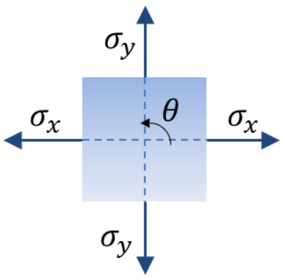
A small segment of material is subjected to plane stresses

$$\sigma_x = 40, \sigma_y = -20 \text{ and } \tau = 30 \text{ N/mm}^2$$

Determine the principal stresses and required angle of rotation to obtain the principal direction by applying

a) Mohr's circle
b) The eigenvalues of the stress tensor (first analytically, then with Matlab)
c) The von Mises reference stress for the original stress state

Ans: $\sigma_1 = 52, \sigma_2 = -32 \text{ N/mm}^2, \theta = 22.5 \text{ deg}$
$\sigma_{ref} = 74.2 \text{ N/mm}^2$



Problem 2

A small segment of material is subjected to plane stresses

$$\sigma_x = 40, \sigma_y = 20 \text{ N/mm}^2$$

Determine the stress state occurring by a 30 deg clockwise rotation of the segment using

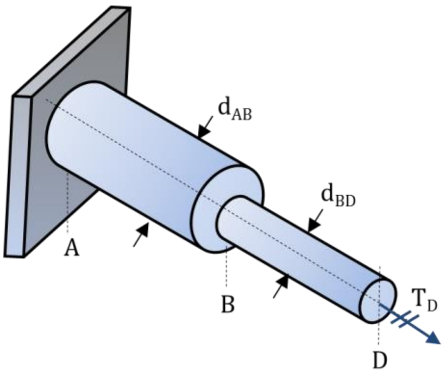
a) Mohr's circle
b) Transformation of the stress tensor

Ans: $\sigma_{x'} = 35, \sigma_{y'} = 25, \tau_{xy'} = -8.7 \text{ N/mm}^2$

Problem 4

Consider the expression for the 3D von Mises reference stress in equation 1 and derive the corresponding expression for

a) A 2D state of stress **Ans**: $\sigma_{ref} = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau}$
b) A 1D state of stress **Ans**: $\sigma_{ref} = \sqrt{\sigma_x^2 + 3\tau}$



Problem 4

A compound cylinder has diameters $d_{AB} = 120$ mm and $d_{BD} = 60$ mm with a transition between the small and large diameter having a fillet with radius $r = 6$ mm. The cylinder is made of machined steel with yield strength $S_y = 355 \text{ N/mm}^2$ and ultimate tensile strength $S_{ut} = 600 \text{ N/mm}^2$.

If a fully reversed cyclic torque $T_D = 1.5$ kNm is applied, determine:

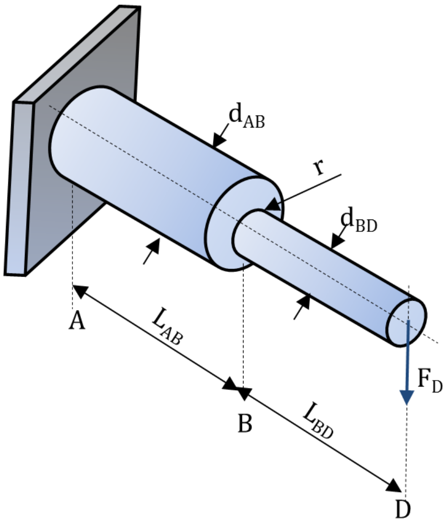
a) The safety factor against static yielding
b) The safety factor against fatigue failure
Assuming infinite lifetime and a reliability of 99.9% corresponding to $C_{rel} = 0.753$ is required by service at ambient temperature
Hint: use von Mises reference stress

If a constant static torque $T_D^{(m)} = 2.5$ kNm is applied in addition to the cyclic load mentioned above, determine:

c) The safety factor against fatigue failure (assuming infinite lifetime is required)

Ans:

a) 4.2,
b) 1.76,
c) Goodman: 1.35
Gerber: 1.66



Problem 5

A compound cylinder has diameters $d_{AB} = 120$ mm and $d_{BD} = 60$ mm with a transition between the small and large diameter having a fillet with radius $r = 3$ mm. The lengths of the segments are given by $L_{AB} = L_{BD} =$ 800 mm The cylinder is made of hot-rolled steel with yield strength $S_y = 235\ \mathrm{N/mm^2}$ and ultimate tensile strength $S_{ut} = 550\ \mathrm{N/mm^2}$. Normal reliability at ambient temperature is required.

If a cyclic and a static end load $F_D^a = F_D^m = 1$ kN is applied, determine:

a) The safety factor against static yielding
b) The safety factor against fatigue failure (assuming infinite lifetime is required)
c) Solve the problem if the loads are redistributed so $F_D^a = 1.5$ and $F_D^m = 0.5$ kN

Ans:

a) 1.35,
b) Goodman; 1.57,
c) a. 1.35, b. 1.14

Problem 6

Reviewing equation 5, derive the equation for the Basquin line (the constants a and b) for

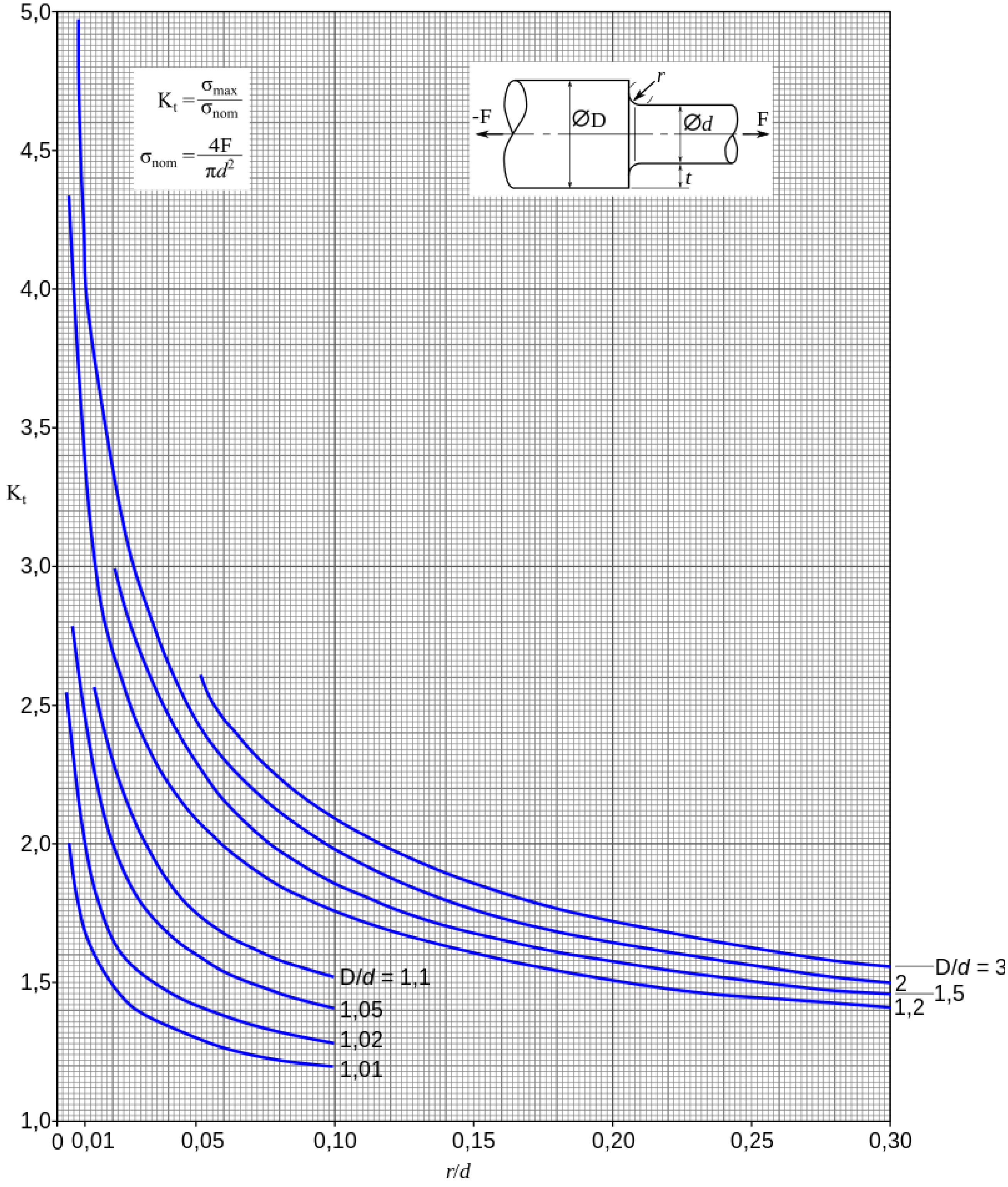
a) The axial load calculated example in the lecture notes
b) Problem 5

Ans: a) $S_e = 174, S_m = 450\,\frac{\mathrm{N}}{\mathrm{mm^2}} \rightarrow a = 1.164 \cdot 10^3, b = -0.1376\,\frac{\mathrm{N}}{\mathrm{mm^2}}$

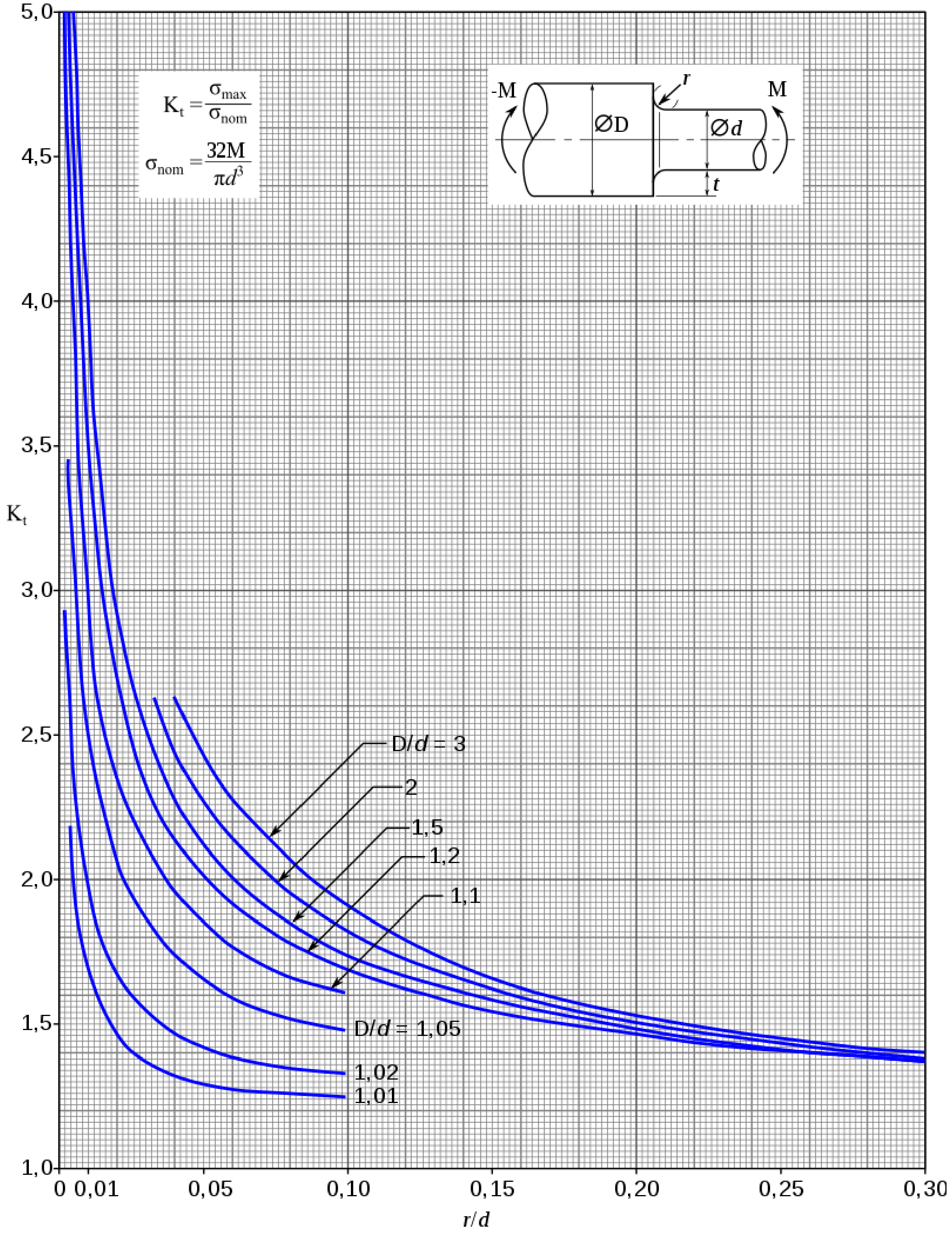
b) $S_e = 153, S_m = 495\,\frac{\mathrm{N}}{\mathrm{mm^2}} \rightarrow a = 1.602 \cdot 10^3, b = -0.1700\,\frac{\mathrm{N}}{\mathrm{mm^2}}$

Appendix A: static stress concentration factors

[Axial loads]

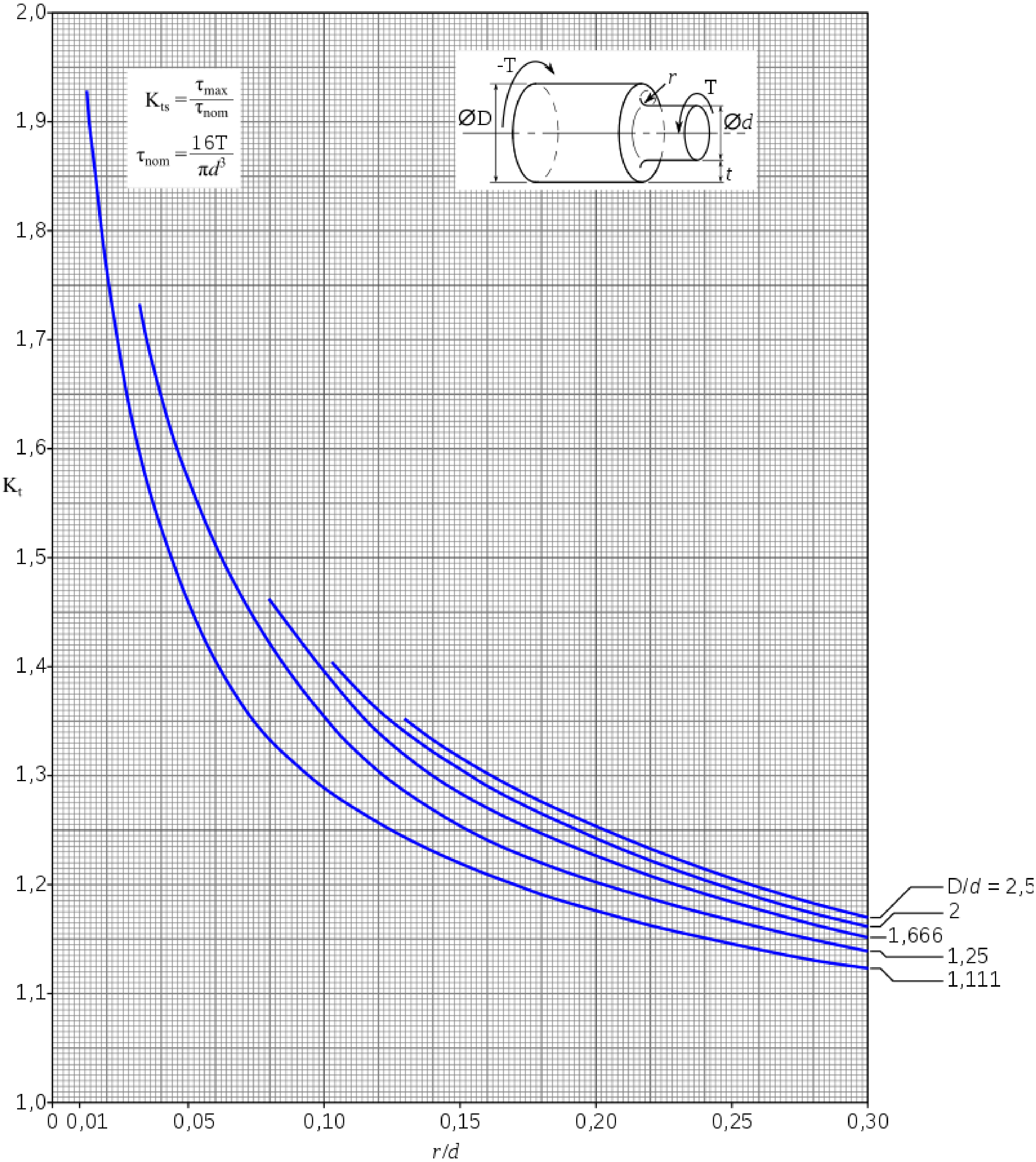


Bending



5,0
4,5
4,0
3,5
3,0
2,5
2,0
1,5
1,0
K_t
$K_t = \frac{\sigma_{max}}{\sigma_{nom}}$
$\sigma_{nom} = \frac{32M}{\pi d^3}$
-M
M
ØD
Ød
r
t
D/d = 3
2
1,5
1,2
1,1
D/d = 1,05
1,02
1,01
0
0,01
0,05
0,10
0,15
0,20
0,25
0,30
r/d

Torsion



$K_{ts} = \frac{\tau_{max}}{\tau_{nom}}$
$\tau_{nom} = \frac{16T}{\pi d^3}$
-T
r
T
ØD
Ød
t
Kt
2,0
1,9
1,8
1,7
1,6
1,5
1,4
1,3
1,2
1,1
1,0
0 0,01
0,05
0,10
0,15
0,20
0,25
0,30
r/d
D/d = 2,5
2
1,666
1,25
1,111