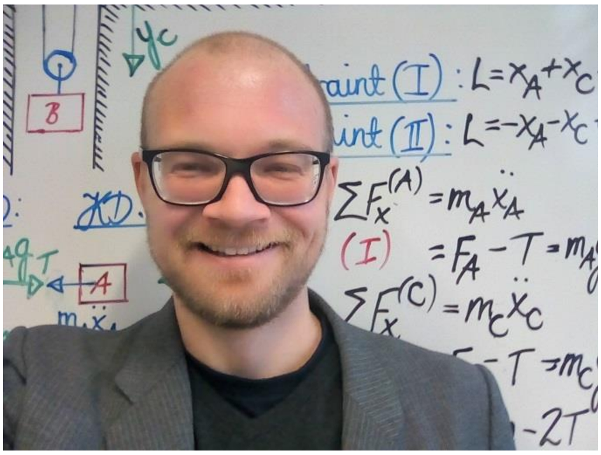
Contents

1. Who we are and what we do: Introduction to Newtonian particle dynamics

1.1. The usual legal stuff

First of all, we as usual have to summarize the legal formalities.



Regards,

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1.2. Basic Definitions – this is where things already get important

Before proceeding with the actual theory, we'll define a few terms of immense importance, since these will be used again and again. Our objective here is to consider the dynamics of particles. While the term *particle* often is used in everyday language when referring to something very small (it's probably the physicists' fault), objects that can be described mathematically as particles can be even very large. One possible definition of the term goes like this:

> *An object for which all properties (for example mass, volume, displacement, velocity, acceleration) can be assigned to a single point*

This implies that phenomena like rotations are not considered. Actually, a lot of the mechanics we will be considering was originally developed by Newton (more about that guy in chapter 3) to describe the planetary motions – and planets are not something an engineer would call small.
We will distinguish between *kinematics* and *kinetics* – also often called *dynamics*. While kinematics only concerns the study of *the geometry of motion*, meaning the relation between positions, velocities and accelerations, kinetics/dynamics in addition is the study of *the cause of motion*, i.e. forces and masses. The latter mentioned discipline can be applied for either *forward dynamics* analysis, by which the motion generated by a set of specified forces acting on an object is determined, and *inverse dynamics*, where the forces required to generate a prescribed motion, are calculated. In each case, our analysis will lead to one or more differential equations, which occasionally (when we're lucky) are reduced to algebraic equations.

After having spent the first year in engineering school doing statics and strength of materials, everybody should by now have grown accustomed to, that all bodies exhibit elastic behavior (bodies are deformable) and are in static equilibrium (meaning that these do not move). This aspect of mechanics will in the current course be turned upside down, since all considered bodies in this course are perfectly rigid, and the only elastic effects we will ever consider are springs, which always will be drawn as such. However, these rigid bodies are now going to move around, meaning that static equilibrium is no longer fulfilled. Please do keep in mind that this is more tricky than it sounds, since students in experience during first year's studies intuitively have gotten used to balancing forces and having those add up to zero. Therefore, we will study how to balance external- and inertial forces in a systematic way in order for us to do this right every time we need to.

1.3. What we're going to learn here (on ultra-short form)

The principles, which we are going to learn to master flawlessly for engineering analysis, will briefly be summarized in the section. The application of those are in general often a bit more tricky than it initially seems, which is why this first part of dynamics to a high extend will be focused not only on theoretical principles, but also a few simple schemes we may apply in order to protect ourselves from ourselves. While students often feel that these schemes limit their individualism and personal freedom (and are quite tedious), it is strongly advised to use these. Even experienced engineers tend to make mistakes, when trying to free-style their way around doing this systematically.

The first principle, which will be introduced, is based only on the kinematics of particles, and relates the motion of two particles, A and B in terms of the relative position between the two. We will refer to this relative term, A/B, as A seen from B, leaving us with the *principle of relative motion* on the form

$$\boldsymbol{r}_A = \boldsymbol{r}_B + \boldsymbol{r}_{A/B}$$

We may differentiate the equation above and apply the principle of relative motion at velocity or acceleration level, and we will need the latter very often when doing rigid body dynamics at the end of this course. However, in order to apply the principle correctly, we will see that it is extremely useful to draw the directions and magnitudes in a table below the vector equation before even thinking about solving it.

Once we have made it past kinematics of particles, we will (re-)introduce Newton's 2nd law, probably the most important equation in engineering at all. This particular equation, simply states that the sum of forces acting on a body equals mass times acceleration measured in the center of gravity, though Newton originally formulated the $m\boldsymbol{a}$-term as the time derivative of the linear momentum, $\boldsymbol{L} = m\boldsymbol{v}$

$$\sum \boldsymbol{F} = m\boldsymbol{a} = \frac{d\boldsymbol{L}}{dt}$$

The term $\sum \boldsymbol{F}$ still denotes the sum of external forces, while the vector $m\boldsymbol{a}$ will be denoted *the inertial forces*. This actually is a whole lot more elegant than it sounds, and tends to get mechanics professors all worked up. The reason for this is that Newton's 2nd law relates forces to kinematic properties and enables us to derive differential equations describing how bodies will move. We call the solutions to these *equations of motion* (EoM). However, this also introduces an additional complexity, which we never had to face while working with static equilibrium (where the right hand side always is zero): in order for this to work, forces and kinematics must be measured in the same coordinate system. This turns out to be harder to ensure intuitively than one would imagine and is among the by far most common sources for wrong signs in equations. In order not to fall for this, we will develop a method, where we chose a coordinate system as reference, draw a free-body diagram (FKD) containing external forces (business as usual) and a kinetic diagram containing the inertial forces (right hand terms) positive in the coordinate directions before applying Newton's 2nd law to derive EoM's. This is going to be a rather big deal and is rater great.

Newton's 2nd law is generally valid and does not require geometrical linearity in order to apply, like most items in the SoM-toolbox would. There's actually only one limitation of great importance: on the form shown above, Newton's 2nd law only holds for systems with constant mass. That means, that there's one (pretty awesome) system we cannot do using this equation … have you guessed it already ? (think megahard!). Right – a rocket, since those ejects mass. For those, the so-called Tsiolkovsky equation is required – a name of great importance, which your mechanics professor unfortunately never has learned to pronounce in a just remotely correct fashion. Therefore, we often simply call it the rocket equation.
Eventually, this can in addition be used to define the angular momentum as the moment of the linear momentum around a given point. It can be shown that the time derivative of the angular momentum gives us the sum of moments

$$\boldsymbol{H}_O = \boldsymbol{r} \times m\boldsymbol{v} \rightarrow \dot{\boldsymbol{H}}_0 = \sum \boldsymbol{M}$$

Finally, Newton's 2nd law may be applied for derivation of EoM's for oscillating systems. For a mass-spring-damper system subjected to a harmonic excitation force, the dynamic equilibrium will be on the form

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{1}{m}F(t)$$
$$= \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}F(t)$$

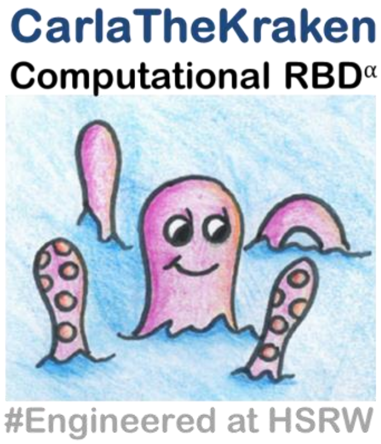
This enables us to derive an immensely useful toolbox for analysis of systems subjected to mechanical vibrations, that can be extended to systems with more than one degree of freedom (DoF) using linear algebra.
I you didn't understand all of this after reading this section, it's alright. That's what the rest of the lectures notes are for. This should just give you an idea of what you have coming.

1.4. Intention and prerequisites

For further readings, the HSRW mechanics professors usually recommend Beer & Johnstons textbook Dynamics (SI-version), which the library has quite a few copies of. However, it is the objective of the current lectures notes to provide a fulfilling theoretical basis with sufficient calculated examples, so the course can be passed without the text book. It must be emphasized, that the ability to study from books rather than videos and slides remains a crucial engineering skill, and that video material cannot replace books and lecture notes (or to rephrase #FightEvilReadBooks).
It is recommended to use the non-programmable calculator, you have been applying for all other exams, for the course and to make friends with it and to learn how to use it properly and reliably. Though this sounds trivial, a calculator is an electronic tool, which requires the user to grow familiar with it's functionalities and idiosyncrasies. This is done most efficiently by practicing this in all subjects when doing exercises using the same calculator.
In the current class, scientific programming (we'll use MatLab) will be used often to obtain numerical solutions to the derived differential equations. While the application of coding in these notes might be brutal and little elegant in it's structure from a computer science perspective, it is actually not the objective to write functioning elegant software, but simply to apply scientific programming as a tool for obtaining solutions. I.e. we are more concerned about numerical hacking than by software development. While scientific programming will be required in the electives on mechanics, it will not be needed to pass the finals in this course. This often leads to the common complaint that this could be left out if it's not needed for the exam. However, the course is not only about passing exams, but also about supplying students with the tools needed for engineering in the future. Furthermore, including code enables us not only to interface the course materials to maths and physics, but also to controls and general numerical modeling. So no matter how frustrated some students get by actually having code in the notes, it's a clever move to learn to solve numerical problems by programming – this is amazingly useful in many engineering disciplines though programming admittedly does have a steep

learning curve when solving problems in this way for the first time. It is recommended to start out simply by running the included codes, and later start modifying them (#Don'tPanic).
The course will require more items from the maths toolbox than strength of materials. More specifically, we will in addition to differential equations in time need vectors and matrices for linear systems of equations and eigenvalue problems (linear algebra). For some reason, it has in the past often occurred that students at the beginning of the course attempt to solve algebraic equations by Gaussian elimination. Though the reasons for this remain unclear, it is highly recommended to use backward substitution in this course due to the simple fast, that students have been practicing this since the 8^{th} or 9^{th} grade, and therefore have way more experience in using this technique securely.

1.5. The Kraken, what she's about and what she's doing here

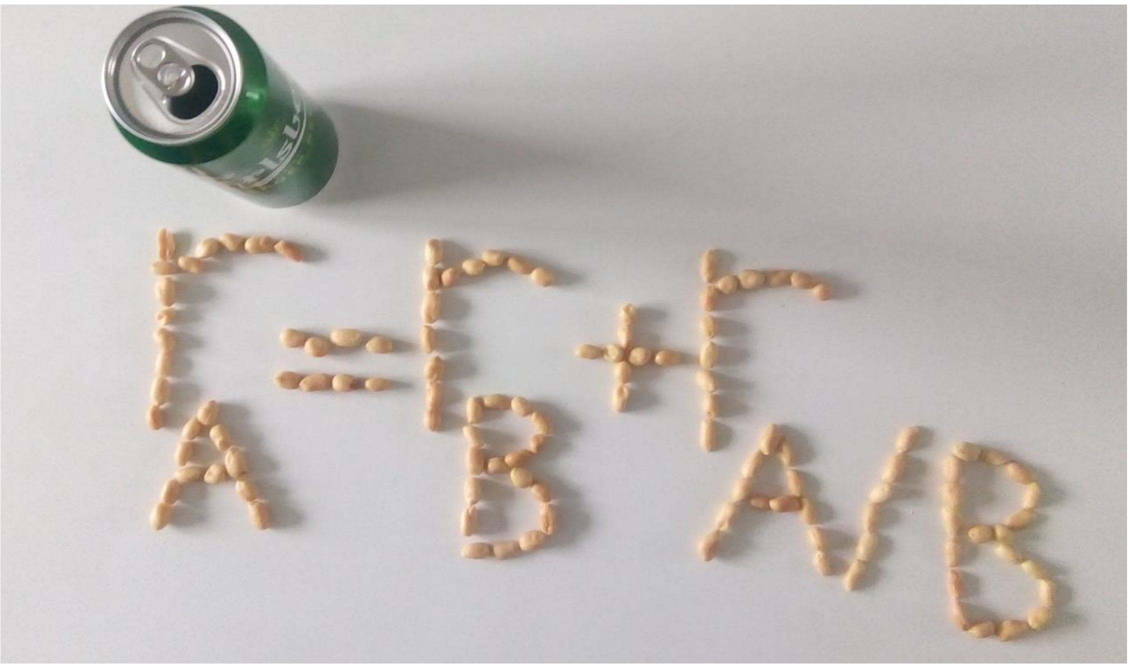


Carla the kraken[1] is your mechanics professor's research/hobby project (and virtual pet). The objective of this project is the development of an engine for computational dynamics and quite a few students have worked with- and on the code in BSc and ARP-projects[2]. The code is (optimistically speaking) available in a Matlab-based alpha version[3], but will at some point in the future be translated to Python or a similar open source based language. The name started as a joke, but actually contains the point, that we're dealing with a code written by engineers for engineers, and from a computer science perspective therefore is and probably will remain a monster[4]. Since the current 2D version in development fits what we are going to do here pretty well, examples of codes will be included in these notes. If you feel like it, you can download it and have fun with it, but these code segments will not be on your finals.

[1] The Kraken doesn't really have a manual yet, but thse theoretical background is described here
[2] There's a YouTube video here with preliminary results
[3] You can download the code here
[4] The Kraken actually had a web-comic buried in the deep abyss of the internet. However, your professor's daughter did apply her mighty coloring skills to make this look good in the name of engineering awesomeness

2. Particle kinematics

In this chapter, the mutual dependencies between positions, velocities and accelerations will be considered. We recall that this discipline is called *kinematics* and refers to *geometry of motion* without any consideration of *cause of motion*. As a consequence, no dependencies of mass and force will be present in the derived equations. Initially, motion of particles in Cartesian (x-y-based) coordinate systems will be recapped, before these are applied in order to introduce the principle of relative motion and the concept of kinematic constrains (dependencies between the motions of two or more particles). Finally, two additional coordinate systems will be introduced, namely a polar and a curvilinear frame. These will turn out to be amazingly useful when calculating accelerations for many problems. Furthermore, this will be needed for rigid body dynamics at the end of the course (and in particular the centripetal acceleration will come back and bite us)



The principle of relative motion is probably the most important result in this chapter. The graphic display shown above was chosen with means that hopefully will make you feel enthusiastic about this.

2.1. 1D-motions

Initially, we will recap the motion of a particle in 1-dimension. This should be pretty straight forward, so we'll keep it short.

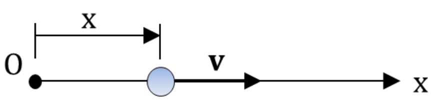


Figure 1 Translation of a particle along a straight line

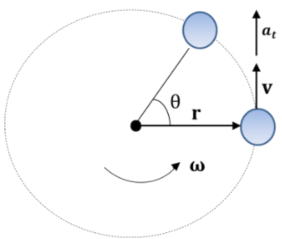


Figure 2 Rotation of a particle along a circular trajectory

2.1.1. Translating motions

For a particle moving along a straight line as shown in Figure 1, the position of the particle is defined as a time function *x(t)* with respect to a fixed origin *O*. It is recalled from basic physics, that the particle velocity is defined as the time derivative of the position

$$v(t) = \frac{dx}{dt} \qquad \left[\frac{\mathrm{m}}{\mathrm{s}}\right] \tag{1.}$$

In addition, the acceleration is defined as the time derivative of the velocity

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} \left[\frac{\mathrm{m}}{\mathrm{s}^2}\right] \tag{2.}$$

For a motion with constant acceleration a and initial conditions v_0 and x_0, we obtain the well-known relations

$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0 \tag{3.}$$

$$v(t) = at + v_0 \tag{4.}$$

In dynamics, it's for convenience common to apply dot-differentiation notation and write $\mathrm{v} = \dot{x}$ $a = \dot{v} = \ddot{x}$, since this form is more compact.

2.1.2. Rotating motions

If a particle rotating along a circular trajectory as shown in Figure 2, the position is defined as an angle $\theta(t)$ with respect to a fixed entity. For such a motion, the principles applying in the previous section may also be applied, and we define a velocity measure in terms of angle as

$$\omega(t) = \dot{\theta} = \frac{d\theta}{dt} \left[\frac{\mathrm{rad}}{\mathrm{s}}\right] \tag{5.}$$

This terms is called the angular velocity and is a measure for how large angle the particle rotates per time unit. In a similar fashion, we can define the angular acceleration

$$\alpha(t) = \frac{d\omega}{dt} = \ddot{\theta} \left[\frac{\mathrm{rad}}{\mathrm{s}^2}\right] \tag{6.}$$

For constant angular accelerations α and initial conditions ω_0 and θ_0, we have

$$\omega(t) = \alpha t + \omega_0 \tag{7.}$$

$$\theta(t) = \frac{1}{2}\alpha t^2 + \omega_0 t + \theta_0 \tag{8.}$$

If required, the angular and linear velocities for rotating motions are related by $v = \omega r$. In a similar fashion, the tangential acceleration and the angular acceleration is related by $a_t = \alpha r$. However, a tangential acceleration is insufficient to sustain the circular trajectory of the particle, and a normal acceleration towards the center of rotation is required as well, since the particle in absence of a such would just move along a straight line. We will consider this in detail in section 0.

2.2. Motion in 2- or 3D in Cartesian coordinate systems

The principles for describing 1D motions in section 2.1.1 can be extended directly in order to describe a motion of a particle in 2- or 3D. For particle moving along a curve described by ***r****(t)*, see Figure 3, the velocity is obtained by

$$v = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} = \frac{d\boldsymbol{r}}{dt} \tag{9.}$$

While the speed is obtained on basis of the arc length *s* along the curve

$$\|v\| = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{s}}{\Delta t} = \frac{d\boldsymbol{s}}{dt} \tag{10.}$$

Equivalently, we may define the instantaneous acceleration as

$$\boldsymbol{a} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{d\boldsymbol{v}}{dt}$$

It is noted, that the kinematic properties, position, velocity and acceleration, now are vectors which have time functions as entries. It is noted, that the total velocity always is tangent to the trajectory. In order to figure out how this works in detail, we will do a calculated example.

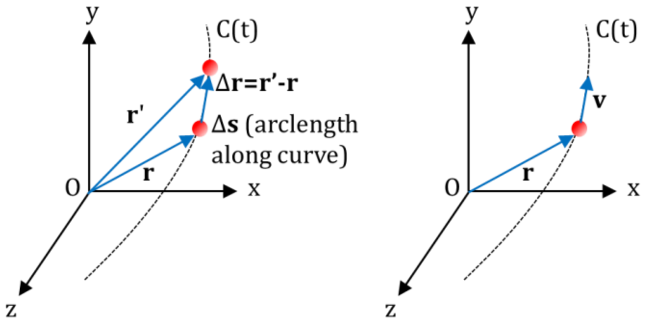


Figure 3 Particle moving along a spatial trajectory

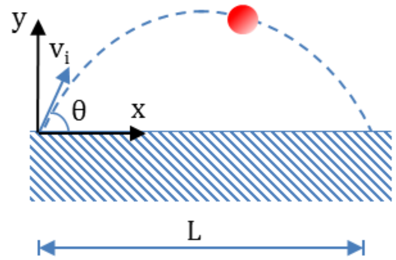


Figure 4 Ballistic curve of a basketball

2.2.1. Calculated example: maximizing the length you can throw a basketball

Problem: Determine the launch angle θ that maximizes the length L for a particle in plane curvilinear motion in the field of gravity fired with initial velocity v_0, see Figure 4.
Solution: Using a cartesian coordinate system in which the particle is fired from the origin, the position functions in the coordinate directions are given by

$$\boldsymbol{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} v_0 \cos\theta t \\ -\frac{1}{2} g t^2 + v_0 \sin\theta t \end{pmatrix}$$

Denoting the time of impact t', we note that $r_y = 0$ for $t = 0$ and $t = t'$. Dividing through with t and thereby elimination the trivial solution, we have

$$\frac{r_y}{t} = -\frac{1}{2} g t + v_0 \sin\theta = 0$$

Solving for t now gives us $t' = \frac{2 v_0 \sin\theta}{g}$. Substituting t' into the expression for the horizontal position, we have the length

$$L = r_x(t') = v_0 \cos\theta \left(\frac{2 v_0 \sin\theta}{g}\right) = \frac{2(v_0)^2}{g} \cos\theta \sin\theta$$

Now, in order to find the optimums, we must solve

$$\frac{dr_x}{d\theta} = 0 \rightarrow \frac{2(v_0)^2}{g} \frac{d}{d\theta}[\cos\theta \sin\theta] = 0 \rightarrow \frac{d}{d\theta}[\cos\theta \sin\theta] = 0$$

The derivative is obtained using the product rule

$$\frac{d}{d\theta}[\cos\theta \sin\theta] = \cos\theta \cos\theta + sin\theta(-\sin\theta) = \cos^2\theta - \sin^2\theta \rightarrow \cos\theta = \sin\theta$$

This equation has $\theta = \frac{\pi}{4} = 45$ deg as solution – that is the angle that gives us the largest value of L

Alternatively, we can use MatLab to plot the motion, but if we're lazy, we could also just have the Kraken do it for us (input codes are contained on the following page). The simulated results from the two codes are shown in Figure 5 and Figure 6 and can be observed to correspond to the result calculated above.

```
%Ballistic curve, NHS-HSRW,070720
clc; close all; clear all;
vi=10;                     %Initial velocity
theta=45;                  %Initial angles
g=9.81;                    %Gravitational acceleration
tlim=3;                    %Sim. time
n=1000;                    %Number of time steps

for i=1:n+1;
    t(i)=tlim/(n+1)*i;
    r(i,:)=[cosd(theta)*vi*t(i),-g/2*t(i)^2+sind(theta)*vi*t(i)];
end

figure; plot(r(:,1),r(:,2)); grid on
xlabel('x [m]'); ylabel('y [m]'); axis equal
```

```
%Example model, Ballistics, Kraken input file, NHS-HSRW,070720
vi=10;                    %Initial velocity
theta=[25,35,45,55,65];   %Initial angles
m=1;                      %Mass (doesn't matter here)
IG=1;                     %(Doesn't matter either)
r=0.1;                    %Ball radius

tlim=5;                   %Set simulation time
PltLim=[0,12,0,5];        %Define plot window

PlotCoGs= "on";           %Plot CoGs of bodies
PlotCoGTrcs="on";         %Plot motion traces
Bdy2BdyCts="off";         %Turn of contacts
SolverType="Var";         %Use ODE45 to solve
CstG="on";                %Turn gravity on

for i=1:length(theta)     %Loop through theta
%Generate a circular body for each run-through
IniCnd=[0,0,0,vi*cosd(theta(i)),vi*sind(theta(i)),0];
GenCrcBdy(IniCnd,m,IG,r);
end
```

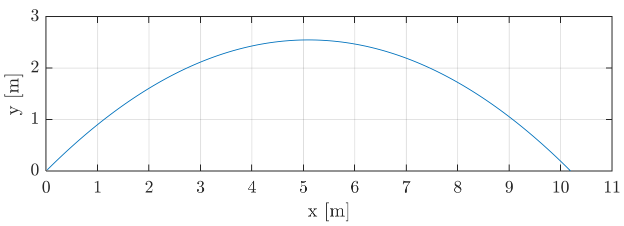


Figure 5 Ballistic curve

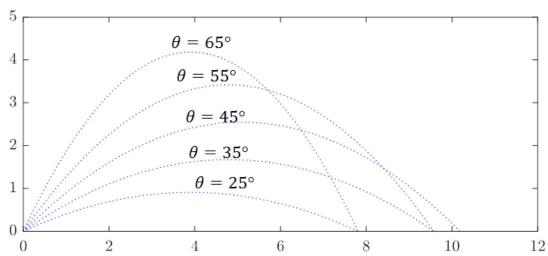


Figure 6 Kraken output

2.3. The concept of kinematic constrains

If the motion of two particles are dependent in a fashion where one particle cannot be moved without this causing a motion of the other, these are said to be kinematically constrained. Examples of two kinematically constrained bodies, are two gear in contact. Any rotation of one gear will lead to a rotation of the other gear. However, if the gears are of different radii, the rate of rotation for the two gears will not be equal. The gear ratio will provide us with a mathematical relation between the two rotations, which constitutes a kinematic constraint. An example of two kinematically constraint particles is shown in the example below.

2.3.1. Calculated example: Mass-wire-pulley systems

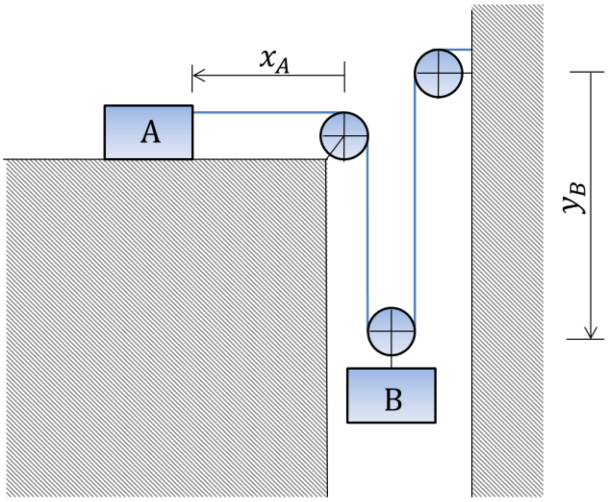


Figure 7 Two masses connected by an inextensible wire through a system of pulleys.

Consider the two blocks A and B connected by a massless inextensible wire guided by a system of massless pulleys as shown in Figure 7. As long as the wire is straight, any change of position of any of the two masses along the axes shown, will cause the other mass to move as well. This means that a kinematic constraint acts between the masses. Due to the arrangement of the wire, both masses will however not move at equal rates. For systems like those, a constraint equation can be obtained by writing an expression for the wire length in terms of the mass positions shown. The wire length is given by

$$x_A + 2y_B + \text{constants} = L \qquad 11.$$

Differentiating with respect to time yields

$$v_A + 2v_B = 0 \qquad 12.$$

$$a_A + 2a_B = 0 \qquad 13.$$

Hence, block A will, if friction is overcome, move and accelerate twice as fast as block B due to the wire arrangement[5]. This type of equations are going to come out very handy in chapter 3.

[5] This type of system has not been implemented properly in the Kraken yet, since wires are a bit more tricky to model than these seem. However, a bright student, Binod Khatri, did implement a chain/belt drive in his BSc thesis (result visualization here)

Bodies may be inter-connected with springs, which would also introduce a coupling between their separate motions. However, effects caused by springs occur due to the presence of elastic forces, which we will account for in the dynamics equilibrium using Newton's 2nd law in chapter 3. Spring effects are therefore *not* to be considered as kinematic constrains. These dependencies in motions between particles can be expressed only in kinematic terms, meaning without ever applying Newton's 2nd law, which is not the case for springs.

2.4. The principle of relative motion

Now to something that is going to turn out to be amazingly useful later (in particular when analyzing the kinematics of rigid bodies). If we consider two particles moving in a Cartesian coordinate system, the position of each can be descripted with respect to the origin using position vectors, see Figure 8. The position of one particle, for example B, can alternatively be expressed with respect to the other particle A. This provides us with the following equation

$$\boldsymbol{r}_B = \boldsymbol{r}_A + \boldsymbol{r}_{B/A} \qquad 14.$$

In which $\boldsymbol{r}_{B/A}$ denotes the position of A relative to- or seen from B. Differentiation provides us with the following expressions:

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A} \rightarrow \boldsymbol{a}_B = \boldsymbol{a}_A + \boldsymbol{a}_{B/A} \qquad 15.$$

These equations are in general known as the principle of relative motion.

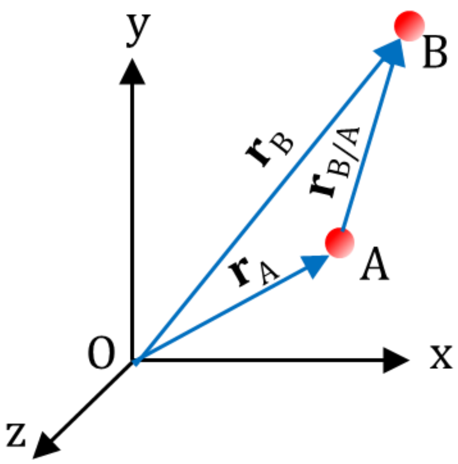


Figure 8 Relative motion of two particles

The developed equations can be solved either geometrically or algebraically (the latter approach is highly recommended). However, both approaches have in common, that the directions of the vectors are moretricky to visualize simultaneously than one would think. Therefore, independent of which solution approach we choose, it is crucial to sketch directions and magnitudes below the principle of relative motion before proceeding. In order to figure out how this exactly is applied for practical problems, we will do a calculated example.

2.4.1. Calculated example: A rocket seen from an airplane

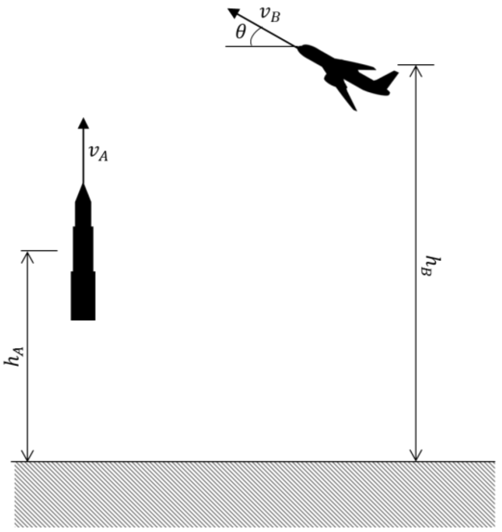


Figure 9

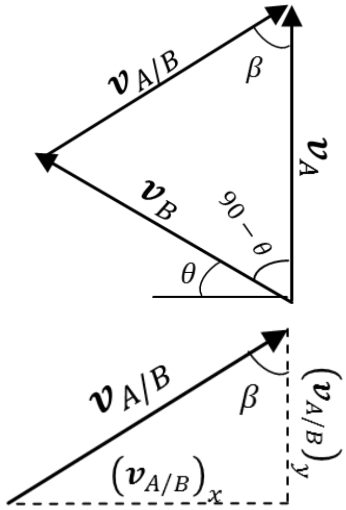


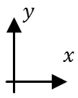
Figure 10

Problem: Determine the velocity of a rocket at the instance after launch where the vertical velocity is $v_A = 850\,\frac{\text{km}}{\text{h}}$[6] as seen from an airplane travelling with velocity $v_B = 1000\,\frac{\text{km}}{\text{h}}$ and climbing at an angle of inclination $\theta = 30$ deg.

Solution: Though not stated explicitly in the problem, the term we require is obviously $v_{A/B}$. Initially, we will convert both velocities to civilized units

$$v_A = 850\,\frac{\text{km}}{\text{h}} = 236.11\,\frac{\text{m}}{\text{s}} \qquad v_B = 1000\,\frac{\text{km}}{\text{h}} = 277.78\,\frac{\text{m}}{\text{s}}$$

And here comes the magic trick: When applying the principle of relative motion as vectorial sum to the two moving bodies, we will always draw a small table containing the known directions and magnitudes:



	$\boldsymbol{v}_A$	$=\ \boldsymbol{v}_B$	$+\ \boldsymbol{v}_{A/B}$
Directions	↑	θ ↖	?
Magnitudes	236.1 m/s	277.8 m/s	?

Having done this (and not a minute before), we may solve the vector equation above in two different ways.

Solution method A: A very old school approach is simply to draw the vectors true to scale, see Figure 10 (top). In the time before pocket calculators, this was convenient, since this allowed for solving relative motion problems by geometrically constructing polygons and measuring the side lengths. In this particular case, the magnitude of the relative velocity is obtained as

$$v_{A/B}^2 = v_A^2 + v_B^2 - 2v_A v_B \cos(90-\theta) \rightarrow v_{A/B} = 296.73\,\frac{\text{m}}{\text{s}}$$

The direction of the relative velocity can be specified both with respect to horizontal or vertical. In the current case, specification with respect to vertical in terms of the ange β is most convenient, and we write

$$\frac{\sin(90-\theta)}{v_{A/B}} = \frac{\sin\beta}{v_B} \rightarrow \beta = 61.61 \text{ deg}$$

[6] It's not like your mechanics professor know that much about rockets except that these are pretty awesome, so I got that value from this video of a SpaceX launch

Solution method B: Rather than going through the tedious process of drawing the vectors as triangles, we could also just consider the two vectors and by projection onto the shown coordinate axes, convert the vector equation to two algebraic equations:

$$\begin{pmatrix} 0 \\ v_A \end{pmatrix} = \begin{pmatrix} -v_B\cos\theta \\ v_B\sin\theta \end{pmatrix} + \boldsymbol{v}_{A/B} \rightarrow \boldsymbol{v}_{A/B} = \begin{pmatrix} (v_{A/B})_x \\ (v_{A/B})_y \end{pmatrix} = \begin{pmatrix} 0 \\ v_A \end{pmatrix} - \begin{pmatrix} -v_B\cos\theta \\ v_B\sin\theta \end{pmatrix} = \begin{pmatrix} v_B\cos\theta \\ v_A - v_B\sin\theta \end{pmatrix} = \begin{pmatrix} 261.02 \\ 141.10 \end{pmatrix} \frac{\text{m}}{\text{s}}$$

Now that wasn't hard at all, and it is highly recommended to apply this approach, since we later in this class will be dealing with geometries which are extremely annoying to draw. In order to compare the two approaches, the vector can be converted to being specified in terms of an angle and a direction:

$$v_{A/B} = \sqrt{(v_{A/B})_x^2 + (v_{A/B})_y^2} = 296.73\,\frac{\text{m}}{\text{s}}$$

$$\beta = \text{atan}\left(\frac{(v_{A/B})_x}{(v_{A/B})_y}\right) = 61.61 \text{ deg}$$

These results can be observed to correspond to what we obtained using solution method A.

So this is the velocity, which the rocket would seem to move with if we were looking at sitting in the airplane.

2.5. Other coordinate frames[7]

This far, all considered motions of particles moving along a given trajectory, have been expressed in Cartesian coordinates with respect to a fixed origin in terms of a position vector $\boldsymbol{r}$ and a velocity $\boldsymbol{v}$ being tangent to the trajectory. This means, that the considered vectors have the usual unit vectors in the coordinate directions, $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ as basis, see Figure 11 (left). However, it is often, in particular when calculating accelerations, convenient to apply a different basis. The following statements can be used as rule of thumb when choosing a basis:

1. If a particle is moving along a circular trajectory, it is most convenient to apply a basis using normal and tangential components (*n-t* components), see Figure 11 (middle). A tangential unit vector $\boldsymbol{e}_t$ and a normal unit vector $\boldsymbol{e}_n$ will be attached to the particle, but the kinematic properties are still calculated referring to a fixed origin in the center of rotation. This basis can theoretically speaking also be applied for a particle moving along a non-circular trajectory, but the origin in the center of rotation would then move in time, which is inconvenient.
2. If a particle is moving along a non-circular curved trajectory, it is often convenient to describe the motion in terms of a position vector $\boldsymbol{r}$ and an angle θ, see Figure 11 (right). This is usually denoted a *polar coordinate system*. The following basis is a radial- and a transverse unit vector, $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$. It is noted, that velocity only will be in the transverse direction $\boldsymbol{e}_\theta$ in the case where a polar frame is applied to describe the motion of a particle along a circular trajectory, and since $\boldsymbol{e}_\theta$ in general contrary to $\boldsymbol{e}_t$ is not tangent

[7] The expressions contained in this section are usually derived during the lectures, and it is the intention to include the derivation in the current section in the future. For now, the derivations will be uploaded in slide form (and if I forget to do so, please remind me).

to the trajectory, a particle would in general have velocity components in both the radial and transverse directions.

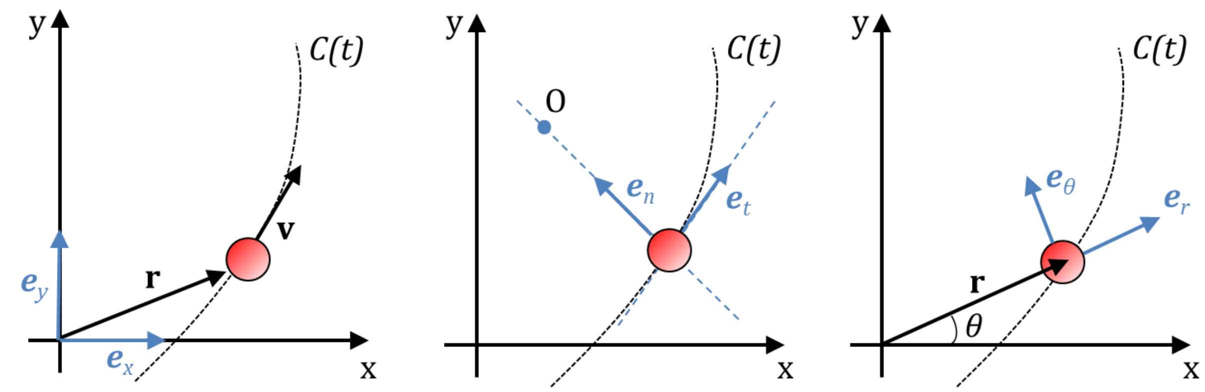


Figure 11 Different choices of basis for a particle moving along a curved trajectory

2.5.1. Normal- and tangential components

When choosing a basis for the motion of a particle based on normal and tangential components as shown in Figure 11 (middle), the velocity is always directed along the unit tangent and can be written as

$$\boldsymbol{v} = v\boldsymbol{e}_t \qquad 16.$$

And now here's something utterly clever. The acceleration can be expressed in terms of a tangential term a_t changing the velocity along the trajectory, and a normal term denoted *the centripetal acceleration* a_n providing the curvature of the trajectory. The acceleration can now be written as

$$\boldsymbol{a} = \frac{dv}{dt}\boldsymbol{e}_t + \frac{v^2}{r}\boldsymbol{e}_n \qquad 17.$$

In case of a rotation specified in terms of angular velocity and acceleration, we have in accordance with section 2.1.2, $a_t = \alpha r = \frac{dv}{dt}$ and $a_n = \omega^2 r = \frac{v^2}{r}$. It is noted, that the particle in absence of a normal acceleration component, simply would be moving along a linear trajectory, and that the normal acceleration is directed towards and *not* away from the center of curvature. The acceleration we would feel if travelling along with the particle is denoted the *centrifugal acceleration*, but constitutes a fictive acceleration component felt in accelerated coordinate-systems. Though the acceleration components occurring in such are briefly considered in section 2.5, we will not have time for giving you a full driver's license in those in the current course, and it is important to emphasize, that these frames for now are to be considered dark arts and are not to be used for calculations. For all we know for now, the centripetal acceleration always points inwards towards the center of rotation due to the arguments stated above.

2.5.2. Radial- and transverse components

If we, as shown in Figure 11 (right), apply a polar coordinate system with a radial and a transverse unit vector as basis for the motion of a particle, the velocity vector is given by

$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta \qquad 18.$$

Again, it is noted, that the velocity has components in both the radial and transverse directions, and that the radial term vanishes only in the case where r is constant, yielding a circular trajectory, since the distance to the fixed origin is constant. The acceleration can be expressed as

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta \qquad 19.$$

In order to apply these expressions, the time derivatives of the r and θ are required. Though in particular the acceleration expression above looks hostile, this is surprisingly easy to apply.

2.5.3. Calculated example: A particle moving along a circular trajectory

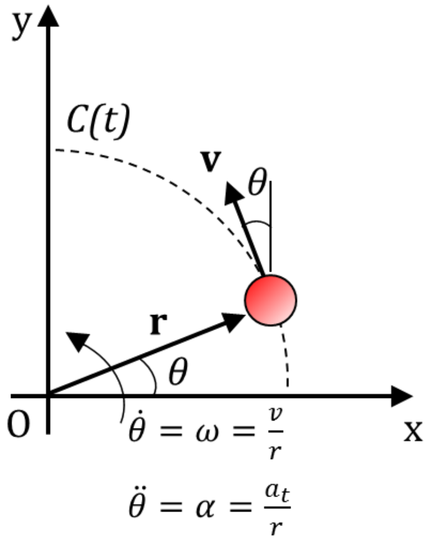


Figure 12 Particle moving along a circular trajectory

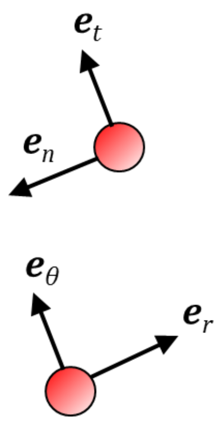


Figure 13 The required unit vectors

Problem: A particle[8] accelerating along a circular trajectory will be considered, see Figure 12. At the instance shown, the velocity and tangential acceleration for the fixed radius are given by

$$v = 5\frac{m}{s} \qquad \frac{dv}{dt} = a_t = 3\frac{m}{s^2} \qquad r = 10m$$

Determine the acceleration of the particle using a) n-t components, b) r-θ components

Solution:

a) Applying normal and tangential components as basis, the acceleration can be calculated by applying equation 17 directly. This is straight forward and yields

$$\boldsymbol{a} = \frac{dv}{dt}\boldsymbol{e}_t + \frac{v^2}{r}\boldsymbol{e}_n = 3\frac{m}{s^2}\boldsymbol{e}_t + \frac{\left(5\frac{m}{s}\right)^2}{10\ m}\boldsymbol{e}_n = 3\frac{m}{s^2}\boldsymbol{e}_t + 2.5\frac{m}{s^2}\boldsymbol{e}_n$$

b) Now redoing the calculation above using radial and transverse components, we will apply equation 19. For a circular trajectory, the following is given

$$\ddot{r}=\dot{r} = 0 \qquad \dot{\theta} = \omega = \frac{v}{r} = 0.5\frac{rad}{s} \qquad \ddot{\theta} = \frac{a_t}{r} = 0.3\frac{rad}{s^2}$$

The acceleration is now given by

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta = (-r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta})\boldsymbol{e}_\theta$$
$$= \left(-(10m)\left(0.5\frac{rad}{s}\right)^2\right)\boldsymbol{e}_r + \left((10m)\left(0.3\frac{rad}{s^2}\right)\right)\boldsymbol{e}_\theta$$
$$= -2.5\frac{m}{s^2}\boldsymbol{e}_r + 3\frac{m}{s^2}\boldsymbol{e}_\theta$$

It is noted that the only difference between the two results are the different signs on the centripetal acceleration since $e_\theta = e_t$ (only true for a circular trajectory) and $e_n = -e_r$, see Figure 13

[8] I didn't get to work out a story for this particular example, so just imagine we're dealing with your favorite sports car, a skateboard or a Tardis – whatever works for you

2.6. Rotating frame of reference (hard 'dark arts' topic)

Should we actually chose to use a rotating frame to express the kinematic properties of a particle, things do get a whole lot more complex, but will also enable us to express the kinematics, in particular the accelerations, we would feel if travelling along a trajectory on top of an accelerated particle, see Figure 14. If the angle of rotation of the moving frame (*x'y'z'*) is given by $\theta = \Omega t$ we can alternatively to using the basis *ijk* of the static frame (*xyz*), express the position of the particle using the moving frame. This provides the expression

$$\boldsymbol{r} = r_x(t)\boldsymbol{i}' + r_y(t)\boldsymbol{j}' + r_z(t)\boldsymbol{k}' \qquad 20.$$

In order to determine the velocity and acceleration, the expression above must be differentiated. It turns out, that the time derivative of a vector function $\boldsymbol{f}$ can be differentiated in a rotating frame using the expression below, where *r* refers to differentiation with respect to the rotating frame

$$\frac{d}{dt}\boldsymbol{f} = \left[\left(\frac{d}{dt}\right)_{\boldsymbol{r}} + \Omega \times\right]\boldsymbol{f} \qquad 21.$$

This looks kindoff weird, but is so central that it sometimes is referred to as the *basic kinematic operator*. So now you have seen what it looks like. Applying this result to the position function defined above, we obtain the velocity

$$\boldsymbol{v} = \left(\frac{dr}{dt}\right)_{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r} \qquad 22.$$

Diffentiation of this results provides the acceleration

$$\boldsymbol{a} = \left(\frac{d^2\boldsymbol{r}}{dt^2}\right)_{\boldsymbol{r}} + \underbrace{2\boldsymbol{\Omega} \times \boldsymbol{v}_r}_{\text{Coriolis}} + \underbrace{\frac{d\boldsymbol{\Omega}}{dt} \times \boldsymbol{r}}_{\text{Euler}} + \underbrace{\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})}_{\text{Centrifugal}} \qquad 23.$$

This expression contains exactly the terms we would feel if moving along with the particle though these are all fictive in the sense that they do not exist in a static (non-rotating frame). The centrifugal force is felt as pointing outwards away from the center of rotating contrary to the centripetal acceleration. In an equivalent sense, the Euler acceleration is felt opposite the tangential acceleration along the trajectory. The Coriolis acceleration is a last, and particular weird term, that occasionally turns out to be small when this principle is applied for mechanical design (yes – people have done that often in the past). The most commonly used example of Coriolis accelerations are the spiraling motions of the wind systems in the equatorial regions[9].

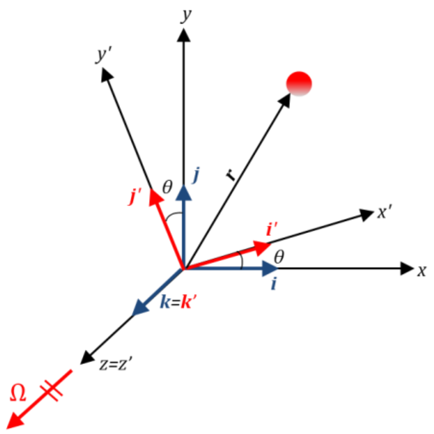


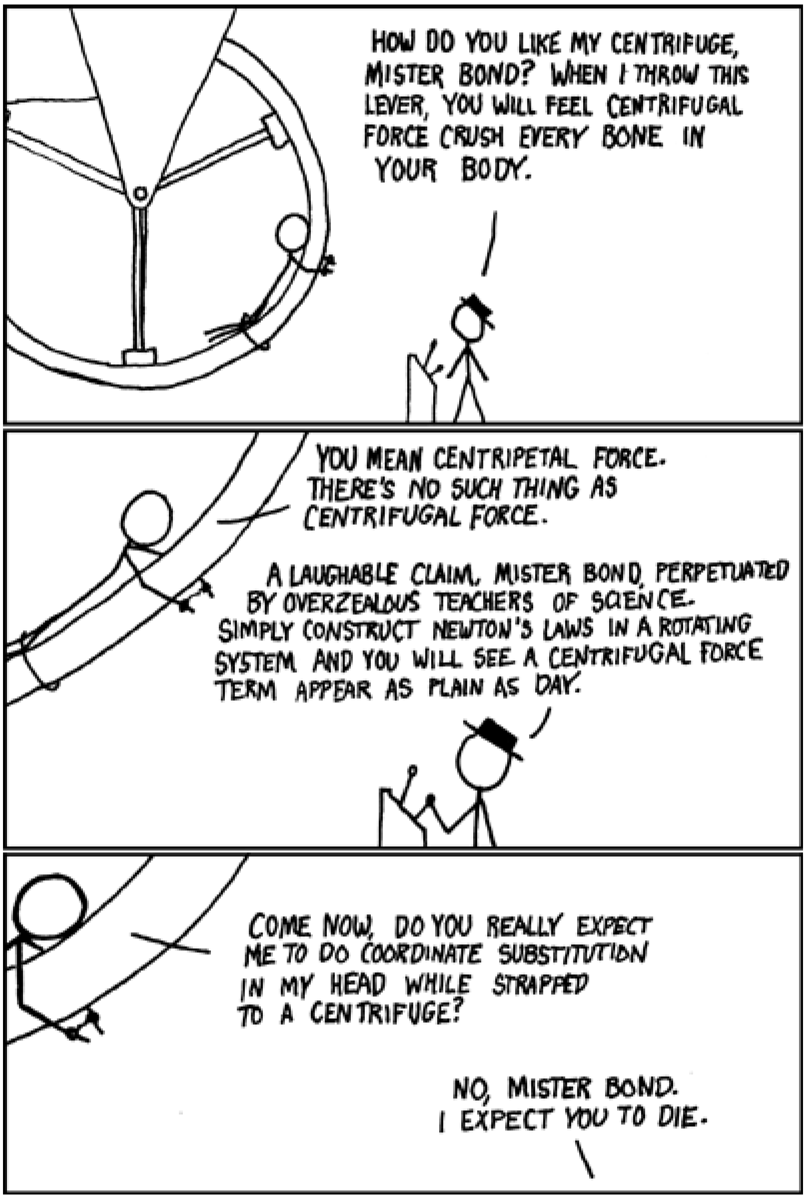
Figure 14 The motion of a particle described in a rotating coordinate frame

This shock summary has as only objective to briefly introduce the relevant terms[10].

[9] The Coriolis acceleration can be observed when throwing a ball in the radial direction inwards or outwards on a merry go around. Due to different velocities of the various points in the radial direction, it will seem as if the ball is accelerated sideways. I.e. it is easier explained on the Discworld than on Earth

[10] Rotating frames will not be on your finals, I promise

Congratulations – you have survived chapter 2. Here's comic from xkcd which in accordance with engineering geek-lore makes fun of people unable to distinguish the centripetal and the centrifugal accelerations.



3. Newton's 2nd Law – kinetics of particles

In this chapter, we will (re-)introduce Newton's 2nd linking the kinematic properties we studied in the previous chapter to the sum of external forces acting on a particle. We will see, that the kinematic properties represented by the accelerations, can be expressed in any of the coordinate frames introduced in section 2.5 without any problems. The challenge, however, is to ensure that the external forces also are expressed in the same frame, and we will work out a problem solving strategy to ensure that this is the case. The solution to the dynamic equilibrium derived from Newton's 2nd law, is called the equations of motion, and these in general are derived to be differential equations. For constant accelerations, these can conveniently be solved as algebraic equations. The theoretical basis is not particular difficult to introduce or comprehend, and as a consequence the current chapter will be very heavy with calculated examples.

Newton's 2nd law – properly the most important equation in mechanics. Actually, Newton developed this approach to mechanics as he by the Royal Society was assigned the task of describing the planetary motions, and in order to solve this problem, he started out by spending a few years developing the infinitesimal calculus. Taking this into account, we can probably consider us lucky, since no modern management would be likely to approving a business case, where the first few years of R&D is used only to develop the mathematical toolbox required to solve the specified problems.

3.1. Formulation and Problem Solving Strategy

Newton's 2nd law is in engineering usually applied on the form stating, that the sum of external forces acting on a particle equals mass times acceleration. On vectorial form, this gives us

$$\sum \boldsymbol{F} = m\boldsymbol{a} \qquad 24.$$

This is actually not the original form, since Newton himself in his bestseller, Principia Mathematica, stated that the sum of external forces equals the time derivate of the linear momentum $\boldsymbol{L} = m\boldsymbol{v}$. This follows from the following derivation

$$\sum F = m\boldsymbol{a} = m\frac{dv}{dt} = \frac{d}{dt}(m\boldsymbol{v}) = \frac{d\boldsymbol{L}}{dt} \qquad 25.$$

Newton's second law may also be expressed as

$$\sum \boldsymbol{F} - m\boldsymbol{a} = 0 \qquad 26.$$

in which $m\boldsymbol{a}$ is denoted the *inertial vector*. The interpretation of this is simply that the inertial vectors may be treated as forces, though these obviously are kinematic properties. Having realized the meaning of this, we will not use the equation on this form, but stick to equation 24.

Newton's 2nd law does not require linearity, and on the form presented in this section it has as only limitation that the particle mass must be constant (more about that in section 4.2).

In order to ensure, that we get the signs right when juggling forces and kinematic properties simultaneously, we will when applying this for problem solving start out by choosing a coordinate system defining our sign convention. We will visualize both external forces and inertial vectors separately, before writing down the dynamic equilibrium to derive equations of motion. This is explained step-by-step in Table 1.

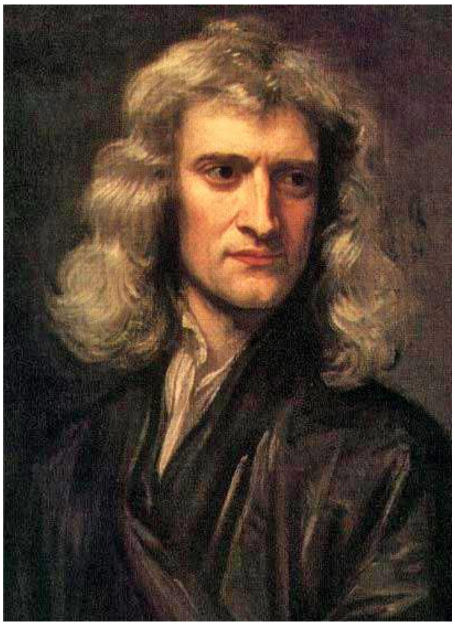


Figure 15 Sir Isaac Newton – the greatest engineer of all times (from Wikimedia – public domain)

Solution strategy

1. Chose a coordinate system (for example a Cartesian, a curvilinear or a polar CSYS)
2. Draw a free body diagram (FBD) of the body that is the subject of your analysis (just like you did in statics)
3. Draw a kinetic (KD) diagram showing the inertial vectors, $\boldsymbol{ma}$, positive in the coordinate directions
4. Formulate the dynamic equilibrium in the directions of the coordinate axes by letting the FBD and KD equal each other – again use the coordinate axes as positive directions
5. If the number of variables is larger than the number of equation, attempt to add kinematic constrains

Table 1 Problem solving strategy

In order to apply this strategy in detail to a problem as simple as possible, we will in the following consider the limit case to statics.

3.1.1. Calculated example

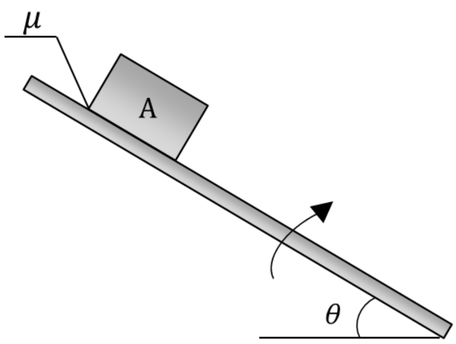


Figure 16 A box on a slowly turning oblique plane

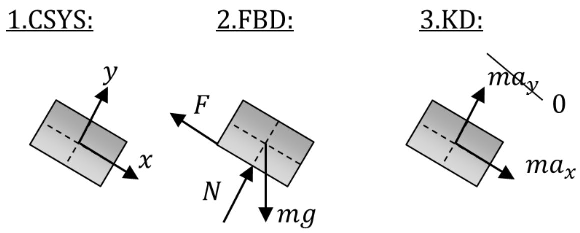


Figure 17 The diagrams required for dynamic equilibrium

Problem: A box on a slowly rotating oblique plane is considered. The coefficient of static friction between box and plane is given by μ. Determine the angle of inclination θ for which slip will occur, and the box will slide downwards.

Solution: Initially, we notice that no mass is specified for the considered box. That could lead us to believe that this is a kinematics problem. However, the box is for low angles restrained against moving by friction, which can only be modeled using forces. Therefore, we will simply have to start out applying the problem solving strategy summed up in Table 1, and hope that something nice happens. Usually, when considering oblique planes, we will use a Cartesian coordinate system with the x-axis in the direction of the plane, see Figure 17 (step 1). Afterwards, we will construct a free-body diagram (FBD) including gravity, friction and normal force, see Figure 17 (step 2). Finally, we draw a kinetic diagram (KD) including the inertial vector positive in the coordinate directions chosen in step 1, see Figure 17 (step 3). We have now graphically visualized the left hand side sum of forces in the FBD, the inertial vector on the right hand side in the KD and all included properties are referring to the sign convention implied when we chose the coordinate system.

What we know about Column friction is, that the frictional force after slip when motion occurs is equal to the dynamic frictional coefficient times the normal force. However, before slip occurs the frictional force will be smaller than or equal to the static frictional coefficient times the normal force. If we did not include the 'smaller than' in that statement, the frictional force would always be equal to $N\mu$ before slip. This means that the frictional force woud be larger than the component of gravity pointing downwards along the incline, and as a consequence the box would accelerate upwards. That doesn't seem right or physically reasonable at all. We should all know this from basic physics. Anyway, if you didn't, don't feel bad, you know it know.

Having gone through step 1-3 in our problem solving strategy, we can now finally go to step 4 and write down the dynamic equilibrium. Applying Newton's 2nd law in the x- and y-direction and realizing that $a_y = 0$, we obtain the following expressions

$$\sum F_x = ma_x = mgsin\theta - F \qquad \sum F_y = ma_y = N - mg\cos\theta = 0$$

Friction is dependent on the normal force N, we have $F \leq N\mu$. For the limit state where slip occurs, this gives us $F = N\mu = mg\cos\theta\mu$ having utilized the equilibrium in the y-direction. For the state just before slip occurs, we actually have $a_x = 0$, and the equilibrium in the x-direction becomes

$$ma_x = mg\sin\theta - mg\cos\theta\mu = 0 \quad \text{(for slip)}$$

$$\rightarrow \mu = \frac{\sin\theta}{\cos\theta} = \tan\theta \rightarrow \theta = \text{atan}(\mu)$$

This turns out to be a quite useful expression to remember, since it provides a simple estimate for when objects on an oblique plane for known static friction will begin to slide downwards.

If you think the previous example was a massive overkill, you may be right – this was mainly to demonstrate how to apply the problem solving strategy to derive a simple useful expression. So let us proceed to something a bit more complex in the next example.

3.1.2. Calculated example: Mass-wire-pulley system

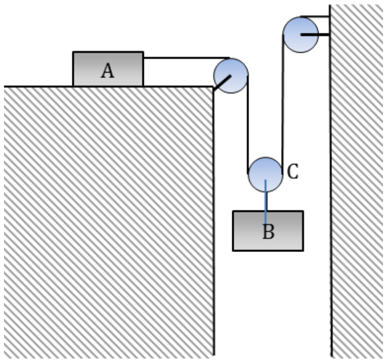


Figure 18 Mass-wire-pulley system

Problem: Two blocks with masses m_A and m_B are connected with a massless inextensible wire by a system of massless pulleys as shown in Figure 18. All contacts in the system are frictionless. Determine the acceleration of both blocks as the system is released from rest.

Solution: Before we get started at all, there is a basic thing we might as well get out of the way. This has in the past made students feel unwell and even physically sick, but there's no way around this, to stay with me now. We introduced the concept of kinematic constrains in section 2.3, and know on this basis that the motion of the blocks is constrained. This realization is due to the fact, that none of the blocks can move without the other block moving. However, if we are going to construct a constraint equation, we will need the axes directions to correspond to the wire length segments required in the wire length equation $L = x_A + 2y_B + cst's$. As a consequence, the axes directions will be chosen as shown in Figure 19 (step 1). We can tell, that block A, when the system is released will slide to the right, meaning in the negative direction – but there is nothing wrong with that. If we had chosen the axes directions opposite, the lengths of the wire segments would not correspond to the positions on the axes, and that would mess up the signs in the following equations. As axes are always defined with a fixed origin, we will in general in systems like this use the pulley centers as such and accept the fact that the considered particle accelerations may come out negative. We may now proceed with FBD's and KD's, see Figure 18.

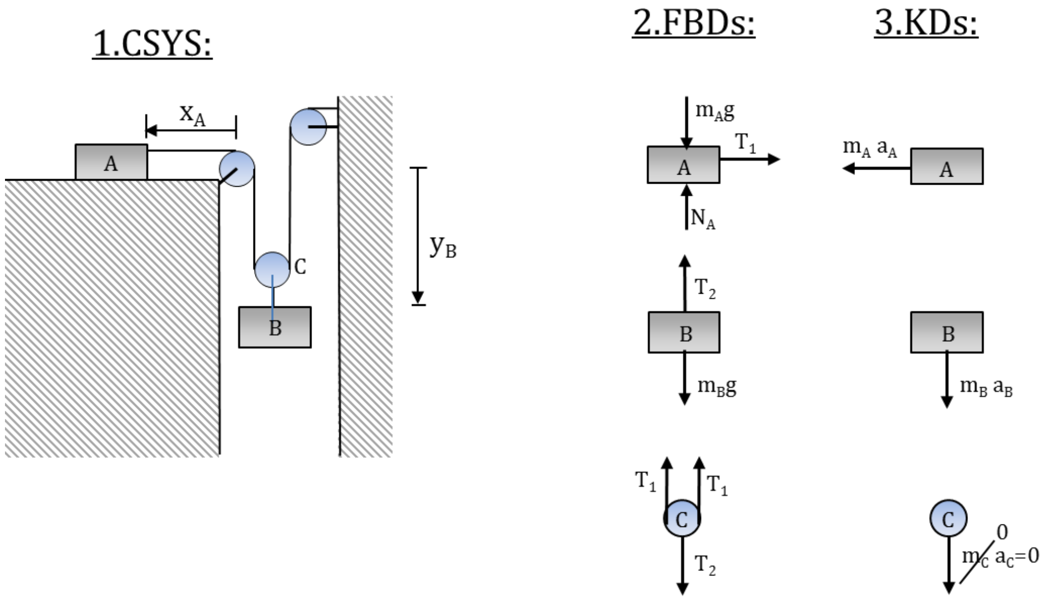


Figure 19 The diagrams required for dynamic analysis

Proceeding to step 4, we can now do the dynamic equilibrium for each particle (pay attention to the signs). We obtain:
Mass A: $m_A\ddot{x}_A = -T_1$
Mass B: $m_B\ddot{y}_B = m_B g - T_2$
Pulley: $T_2 - 2T_1 = 0$ (since the pulley is massless)
Now that (hopefully) went really well, but when counting the unknowns, we realize that that we for three equations of equilibrium have four unknowns m_A, m_B, T_1 and T_2. This is where kinematic constraints turn out to be amazingly useful, since we already have realized that $x_A + 2y_B + cst's = L \rightarrow \ddot{x}_A = -2\ddot{y}_B$. This provides us with a fourth equation enabling us to solve for the unknown acceleration unknowns (Try using backward substitution yourself). We obtain

$$\ddot{y}_B = \frac{m_B g}{m_B + 4m_A} \qquad \ddot{x}_A = -2\ddot{y}_B$$

As expected, the acceleration of block A is negative, since this will move in the negative direction of the chosen axes.
Knowing that $T_2 = 2T_1$ with massless pulleys, it is common and legitimate only to apply one wire tension, and simply add two times T upwards on block B when drawing the free-body diagrams.

Having now done two horribly theoretical examples, we will consider something more applied before continuing. I decided that we will do a train, because trains are cool.

3.1.3. Calculated example: A braking train

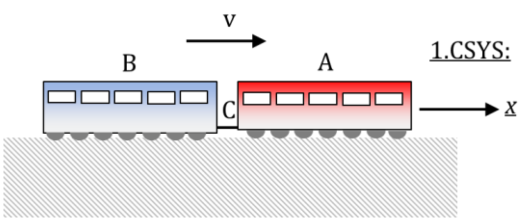


Figure 20 Braking train

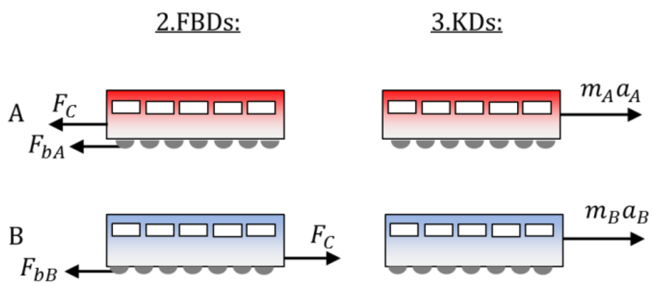


Figure 21 The diagrams required for dynamic equilibrium

Problem: two train wagons with masses m_A=10 mton and m_B=15 mton are driving at a speed of 120 km/h when the brakes are applied. The breaking forces are respectively F_{bA}=50 kN and F_{bB}=30 kN. Determine a) the distance traveled before both wagons are at rest. b) the horisontal force component in the coupling marked C during braking
Solution:
We will chose the positive direction to the right in the direction of the velocity. Having done so, FBD's and KD's can be constructed for each wagon, see Figure 21. Note that we, as always, point the inertial vectors in the positive coordinate directions. The directions of the forces acting in coupling C are of unknown direction, and are therefore assumed.

We can now do the dynamic equilibrium (step 4)

Wagon A: $\sum F_A = m_A a_A = -F_{bA} - F_C$ (I)

Wagon B: $\sum F_B = m_B a_B = -F_{bB} + F_C$ (II)

having two equations with three unknowns, a_A, a_B and F_C. Luckily, it is easy do to a kinematic constraint (step 5) for the system, since the rigid connection between the two wagons will cause those to have the same accelerations, i. e. $a_A = a_B = a$ (III)

Substituting this into equation (I) and (II), we have

$$\text{(I)}: F_C = -F_{bA} - m_A a \qquad \text{(II)}: F_C = m_B a + F_{bB}$$

$$\rightarrow -F_{bA} - m_A a = m_B a + F_{bB} \rightarrow a = -\frac{F_{bA}+F_{bB}}{m_A+m_B} = -3.2\,\frac{m}{s^2}$$

As expected, the calculated acceleration is negative, since the velocity is decreasing. The braking distance can now be calculated using the kinematics of a 1D translating motion

$$x = \frac{1}{2}at^2 + v_0 t \quad \text{(IV)} \qquad v = at + v_0 \quad \text{(V)}$$

with $v = 0 \rightarrow t = -\frac{v_0}{a}$ in equation (V), we obtain the braking distance as

$$x = \frac{1}{2}at^2 + v_0 t = \frac{1}{2}a\left(-\frac{v_0}{a}\right)^2 + v_0\left(-\frac{v_0}{a}\right) = 173.6\,m$$

The coupling force can easily be calculated from equation (II)

$$F_C = m_B a + F_{bB} = -18000\,N = -18\,kN$$

We will later return to this exact problem and solve it again using the principle of work and energy.

This far, we have only considered dynamic equilibrium in Cartesian coordinates. Furthermore, the considered particles have been subjected to constant accelerations, meaning that the equations of motion have been an algebraic rather than a differential equations. In the following calculated example, we will therefore consider a dynamic equilibrium using polar coordinates as basis where the equation of motion actually is on form of a differential equation. This is the point where it really becomes clear, that 1st year math and programming wasn't just (but also) for fun. We are going to need quite a few of these topics now.

3.1.4. Calculated example: The particle pendulum

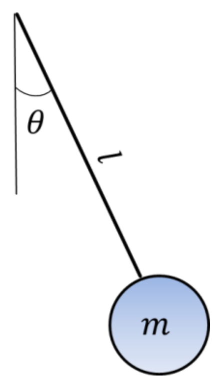


Figure 22 The particle pendulum

Problem: We will now consider a particle with mass m attached to a chord of length $l = 0.5\,m$ allowing this to swing back and forth as pendulum with the angle θ as degree of freedom, see Figure 22. Determine: a) the equation of motion for small angles ($\theta \ll 1$), b) the angular response in time if the pendulum is released with no initial angular velocity $\omega_i = 0$ and an initial angle $\theta_i = 10$ deg, c) the frequency of the motion, d) the pendulum response for large angles

Solution: The particle will due to the chord constraint swing back and forth along a segment of a circular trajectory. As a consequence, a Cartesian coordinate frame is not particular convenient, since we would have to express x and y as functions of $\theta(t)$ anyway. We will apply a radial- and transverse frame instead and draw a FBD and a KD, see Figure 23.

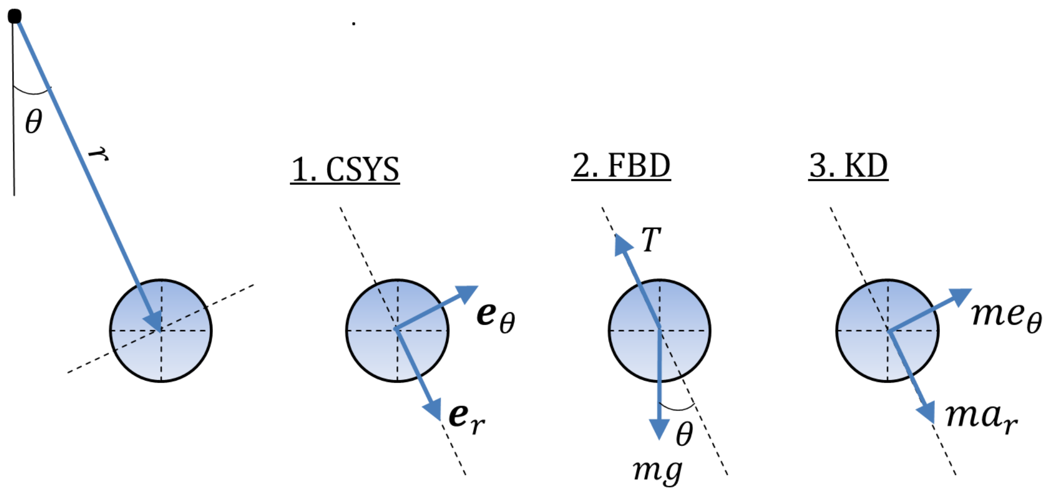


Figure 23 Diagrams required for derivation of the equation of motion for a particle pendulum

Initially, straightening out the kinematics, we recall from section 2.5.2 that the acceleration can be expressed as

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta$$

For a circular trajectory, we have $\ddot{r} = \dot{r} = 0$ and $r = l$, reducing the acceleration expression to

$$\boldsymbol{a} = (-l\dot{\theta}^2)\boldsymbol{e}_r + (l\ddot{\theta})\boldsymbol{e}_\theta$$

We recall the relations $\dot{\theta} = \omega$ and $v = \omega l$, so we have $l\dot{\theta}^2 = l\omega^2 = l\frac{v^2}{l^2} = \frac{v^2}{l}$.

We can now do the dynamic equilibrium in radial and transverse components

$$\sum F_r = ma_r = -ml\dot{\theta}^2 = -T + mg\cos\theta \rightarrow T = ml\dot{\theta}^2 + mg\cos\theta$$

$$\sum F_\theta = ma_\theta = ml\ddot{\theta} = -mg\sin\theta \rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

The first equation, the radial equilibrium, provides an expression for the chord tension, while the second equation, the transverse equilibrium, constitutes the dynamic equilibrium of the particle motion. There's however a problem here: this differential equation is due to the $\sin\theta$ term nonlinear, meaning that it is complex to obtain an analytical solution[11]. But since we are after the response for small angles, we have $\sin\theta \approx \theta$ for $\theta \ll 1$. The dynamic equilibrium becomes

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

We say, that the equation has been linearized. From maths, we know that the solution to this type of linear differential equation is on the form $\theta(t) = C_1\cos(\omega_n t) + C_2\sin(\omega_n t)$ (If you don't believe it, try differentiating this expression twice and plug this in to the equation). The equation of motion is *harmonic*. Also, we remember from maths, that the circular frequency

$$\omega_n = \sqrt{\frac{k}{m}}$$

can be found directly from the differential equation as follows:

[11] There is an analytical solution, it comes on the form of an elliptic integral

$$\mathbf{1}\ddot{\theta} + \underbrace{\frac{\boldsymbol{g}}{\boldsymbol{l}}}_{\omega_n^2}\theta = 0$$

This enables us to calculate the frequency (number of oscillations per second) and the period (duration of one oscillation). What is still left to do is to determine the constants C_1 and C_2, which as usual are found on basis of the initial conditions, the initial angle θ_i, and the initial angular velocity ω_i given that $\theta(0) = \theta_i$ and $\dot{\theta}(0) = \omega_i$. We apply the equation of motion and calculate the angular velocity by time differentiation

$$\theta(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)$$

$$\dot{\theta}(t) = \omega(t) = -\omega_n C_1 \sin(\omega_n t) + \omega_n C_2 \cos(\omega_n t)$$

The initial conditions can now along with $t = 0$ be plugged directly into these two equations providing us with the two following equations

$$\theta_i = C_1 \cos(\omega_n \cdot 0) + C_2 \sin(\omega_n \cdot 0) \rightarrow C_1 = \theta_i$$

$$\omega_i = -\omega_n C_1 \sin(\omega_n \cdot 0) + \omega_n C_2 \cos(\omega_n \cdot 0) \rightarrow C_2 = \frac{\omega_i}{\omega_n}$$

So the constants in the equations have been determined. However, it would be awfully convenient if the equation of motion could be expressed only with a single harmonic term rather than two, since this allows us to recognize the amplitude and phase shift directly. Luckily, we can pull out the following trick (like a rabbit from a hat):

From Maths we know (Lemma ... or something):
A linear combination of sine and cosine functions with equal angular velocity ω can be rewritten as a sine function with a different amplitude A and a phase shift ϕ

$$C_1\, cos(\omega t) + C_2\, sin(\omega t) = A\, sin(\omega t + \phi)$$

with $A = \sqrt{{C_1}^2 + {C_2}^2}$ *and* $tan\phi = \frac{C_1}{C_2}$

Applying this, the equation of motion can be rewritten

$$\theta(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t) = \theta_m \sin(\omega_n t + \varphi)$$

with amplitude and phase shift

$$\theta_m = \sqrt{{C_1}^2 + {C_2}^2} = \sqrt{{\theta_i}^2 + \left(\frac{\omega_i}{\omega_n}\right)^2} \qquad \varphi = \mathrm{atan}\left(\frac{\theta_i \omega_n}{\omega_i}\right)$$

These expressions are generally valid for any initial conditions. However, the initial conditions specified in the problem were $\theta(0) = \theta_i$ and $\dot{\theta}(0) = \omega_i = 0$. Therefore, for the actual case considered, we have

$$\theta_m = \sqrt{{\theta_i}^2 + \left(\frac{0}{\omega_n}\right)^2} = \theta_i = 0.175\, rad \qquad \varphi = \mathrm{atan}\left(\frac{\theta_i \omega_n}{0}\right) = \mathrm{atan}(\infty) = \frac{\pi}{2}$$

$$\omega_n = \sqrt{\frac{g}{l}} \rightarrow f_n = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \rightarrow T_n = 2\pi\sqrt{\frac{l}{g}} = 1.4185 s$$

The pendulum response can based on the equation of motion be plotted, see Figure 24.

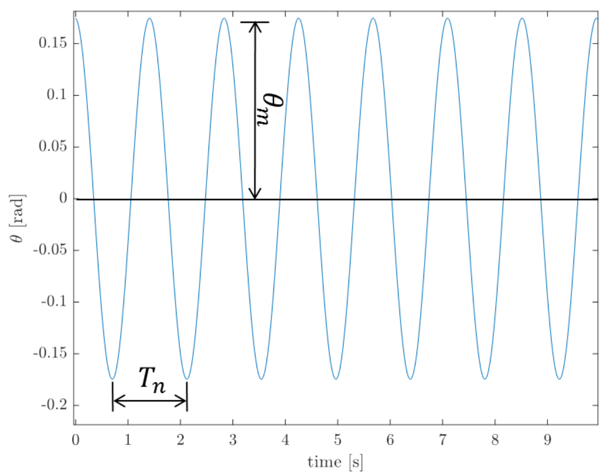


Figure 24 Analytically determined EoM

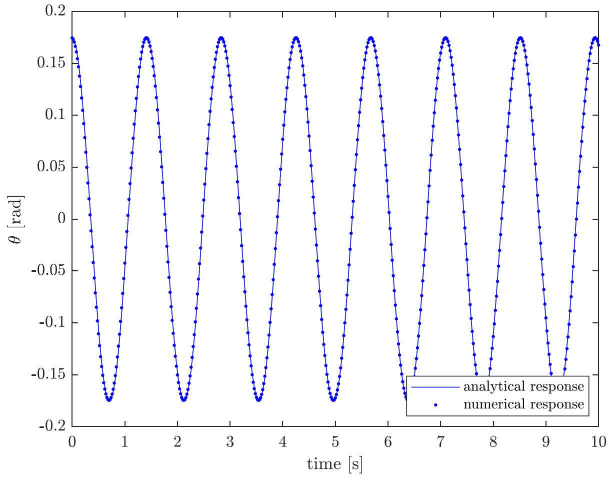


Figure 25 Analytical and numerical solution to the dynamic equilibrium

The last thing that is left to consider is what the solution to the original non-linear differential equation would look like. Rather than obtaining an analytical solution, we will use the opportunity to integrate the dynamic equilibrium numerically. In this context, the simplest possible integrator will be applied, namely the first order Euler scheme. We will start out by isolating the second order term

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \rightarrow \ddot{\theta} = \alpha = -\frac{g}{l}\sin\theta$$

We now rewrite the diffential equations on the form

$$\frac{d\theta}{dt} = \omega \rightarrow \Delta\theta \approx \omega\Delta t \qquad \frac{d\omega}{dt} = \alpha \rightarrow \Delta\omega \approx \alpha\Delta t$$

Using this results, we may approximate the solution to the differential equation for small steps through time by

$$\omega_{i+1} \approx \omega_i + \Delta\omega = \omega_i + \alpha_i\Delta t$$
$$\theta_{i+1} \approx \theta_i + \Delta\theta = \theta_i + \omega_i\Delta t$$

This scheme contrary to higher order integrators has the advantage that it is quite easy to write yourself, if you ever need to solve a non-linear differential equation in time numerically (if you get to work with certain branches of R&D, that will happen like every second day). However, the main disadvantage with the first order numerical scheme shown above, known as the explicit Euler integrator, since only information from the previous time step is used to update the current step, is that it's fairly inaccurate. Actually, the results often are numerical garbage. But a very simple trick increases the accuracy significantly. If we allow us selves to use information from the current time step for the update and start out with the angular velocity, the second step can be written as

$$\theta_{i+1} \approx \theta_i + \Delta\theta = \theta_i + \omega_{i+1}\Delta t$$

This is known as the implicit Euler integrator, and usually works a whole lot better than the original explicit scheme.

So there is a lot of work in oscillating motions[12]. The Matlab code for numerical integration of the pendulum equations is presented on the next page.

[12] Actually, an old dynamics student of mine, Danylo Bolzhelarskyi, did this cool parameterized animation of pendulum waves – which is a very easy trick you can use to impress your muggle friends. This is an example of one of the many systems we intended to build and ended up programming instead.

```
%Particle pendulum, NHS-HSRW, 08.07.20

%Clear workspace, command window and close everything
clc; clear all; close all;
%Define system parameters (self-explanatory)
g=9.81; l=0.5; theta_i=10*pi/180; omega_i=0;
%Define numerical parameters
n=500; tlim=10;
%Setup time vector
for i=1:n+1; t(i)=tlim/n*(i-1); end
dt=t(2)-t(1);
%Initialize vectors, set initial values
theta_num(n+1)=0;        omega_num(n+1)=0;
theta_num(1)=theta_i;    omega_num(1)=omega_i;
theta_ana(n+1)=0;        theta_ana(1)=theta_i;
%Do time integration
for i=2:n+1;
    alpha_num=-g/l*sin(theta_num(i-1));
    omega_num(i)=omega_num(i-1)+dt*alpha_num;
    theta_num(i)=theta_num(i-1)+dt*omega_num(i);
end
%Evaluate analytical response
omega_n=sqrt(g/l);
theta_m=sqrt(theta_i^2+(omega_i/omega_n)^2);
phi=atan(theta_i*omega_n/omega_i);
for i=2:n+1;
    theta_ana(i)=theta_m*sin(omega_n*t(i)+phi);
end
%Plot the solutions
figure; plot(t,theta_ana,'b',t,theta_num,'b.');
legend('analytical response','numerical response')
xlabel('time [s]'); ylabel('$\theta$ [rad]')
```

And here's a fun fact: if you paste the following code into Matlab after running the script above, Matlab will do a little movie of the pendulum.

```
figure
for i=1:n+1
    x(i)=l*sin(theta_num(i)); y(i)=-l*cos(theta_num(i));
    plot(x(i),y(i),'b.','MarkerSize',20)
    hold on
    plot([0 x(i)],[0 y(i)],'b-',[0 x(i)],[0 y(i)],'r-');
    hold off
    axis equal; axis([-l l -l 0]*1.1);
    f(i) = getframe(gcf);
end
```

The last thing we could try to do here, is to convert the angle (or angular acceleration responses) from the time domain to the frequency domain by running these through a Fast Fourier Transformation (FFT). This enables us to see the frequency content of the signal. We already know from the equation of motion, that the only frequency that should be present is the natural frequency given by

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{l}} = 0.71\ Hz$$

The required Matlab code is shown below.

```
dt=n/(tlim);                            %log rate
xdft = fft(theta_num);
xdft = xdft(1:length(theta_num)/2+1freq = linspace(0,dt/2,length(theta_num)/2+1);
figure; plot(freq,abs(xdft),'b'
axis([0 5 0 50])
```

The frequency content is shown in Figure 26 and the peak occurring can be observed to correspond with the natural frequency calculated above.

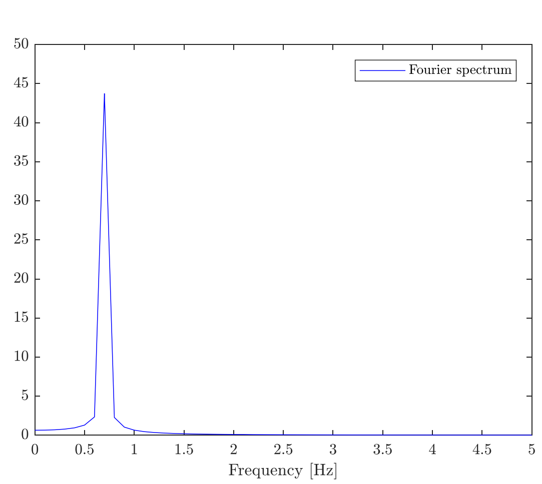


Figure 26 FFT (frequency content) of the pendulum angle response

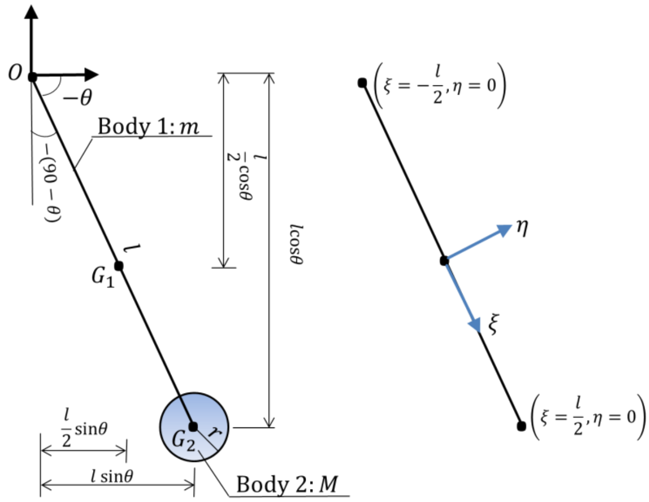


Figure 27 Developing a Kraken model of the pendulum

3.1.5. Calculated example: Particle pendulum with the kraken

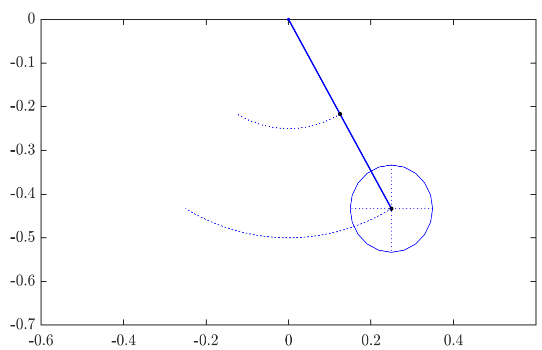


Figure 28 Kraken pendulum model

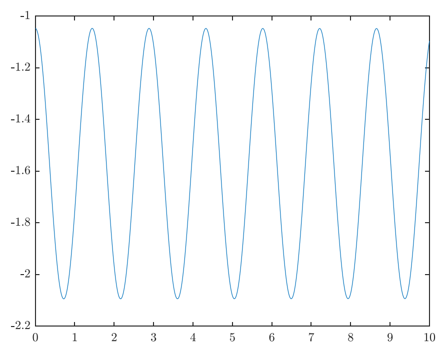


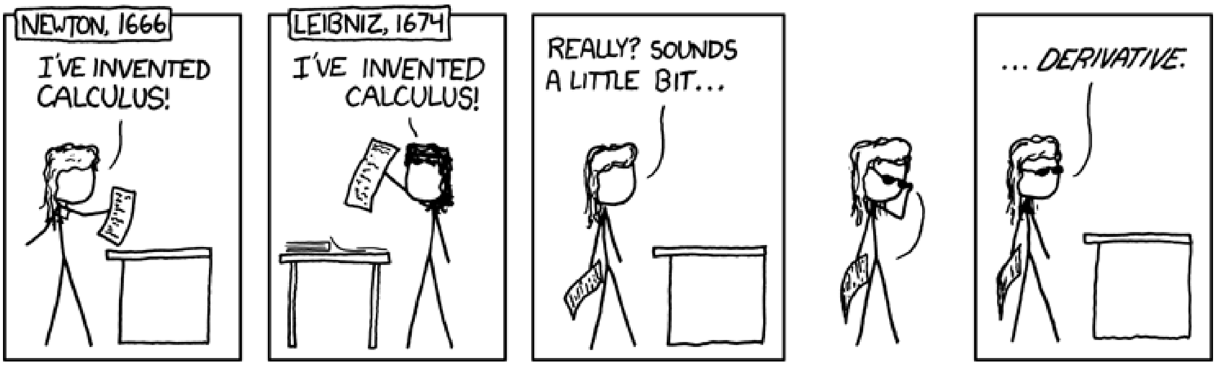
Figure 29 Angular response from analysis

As a lazy alternative to the approach presented in the previous section, we could off course also just use the Kraken do to a pendulum model, see Figure 28. The model configuration is shown in Figure 27. The pendulum chord definition is based on the shown local coordinate system. The obtained angle is shown in Figure 29. The frequency and amplitude can be observed to correspond to the results from the previous analysis. Since the angle of a rigid body in computational dynamics always is measured with respect to the global x-axis and is measured as being positive in the counter-clockwise direction, the calculated angle has an offset with respect to the results in the previous example.

```
%Example model, Particle pendulum, Kraken input file, NHS-
HSRW,080720
%Define system parameters
g=9.81; l=0.5; theta_i=30*pi/180; r=0.1; m=0.01; M=10;
IG1=1/12*m*l^2; IG2=2/5*M*r^2;
%Define numerical parameters
tlim=10;          nt=1000;
%Changes in standard config
Bdb="off";        PlotBdbs= "off";     Bdy2BdyCts="off";
Bdy2WldCts="off"; SolverType="Var";
%Set plot window size
PltLim=[-0.6,0.6,-0.7,0];
%Generate pendulum chord as rigid line body
x1=l/2*sin(theta_i);   y1=-l/2*cos(theta_i);
IniCnd=[x1,y1,-(pi/2-theta_i),0,0,0];
GenLnsBdy(IniCnd,m,IG1,l);
%Generate pendulum point mass as circular body
x2=l*sin(theta_i);   y2=-l*cos(theta_i);
IniCnd=[x2,y2,0,0,0,0];
GenCrcBdy(IniCnd,M,1/2*M*r^2,r);
%Pin upper end of chord to world
Bdy2WldPinJnt(1,[-l/2,0])
%Pin point mass to chord
Bdy2BdyPinJnt(1,[l/2,0],2,[0,0])
%Plot chord angle
PltBdyRsp(1,3)
```

Actually, you have just been tricked to do the analysis of a vibrating system – if this confused you, we will return to those in chapter 5.

You have survived chapter 3. Congratulations.



4. Angular momentum, asteroids and a rocket

Most of contents in this chapter could have been introduced in chapter 3, since we still will be studying Newton's 2nd law and it's applications – in particular planetary orbits. The original intend, when preparing this course, was to derive the theory properly and in detail, since this actually was the problem Isaac Newton aimed to solve as he developed the theory behind mechanics. However, it almost immediately became clear, that this would require an entire semester – and no matter how interesting this is, we also have other topics to study. Therefore, the angular momentum and it's main properties will be introduced, but we will only sum up the analytical solution to the orbital mechanics Newton came up with. Instead of deriving this, we will show that the problem solving strategy from the previous chapter can be applied for solving this problem in Cartesian coordinates, though the obtained solutions are not on a particular pretty form. Finally, the so-called rocket equation will be considered, since Newton's 2nd law on the original form is not valid for systems with changing mass.

We will in the following section define the angular momentum as the moment of the linear momentum. This quantity is constant for a particle moving under a central force. Furthermore, the time derivative of the angular momentum gives us the sum of moments, which will we are going to need later to describe the rotational equilibrium of rigid bodies.

4.1. The angular momentum

Newton's 2nd law was originally formulated as the resulting force equal to the time derivative of the linear momentum L

$$\sum F = m\boldsymbol{a} = m\frac{dv}{dt} = \frac{d}{dt}(m\boldsymbol{v}) = \frac{d\boldsymbol{L}}{dt}$$

It follows that the linear momentum is constant in both magnitude and direction if the sum of forces is 0. The angular momentum H around the fixed point O is now defined as the moment of the momentum

$$\boldsymbol{H}_O = \boldsymbol{r} \times m\boldsymbol{v} = \boldsymbol{r} \times \boldsymbol{L} \qquad 27.$$

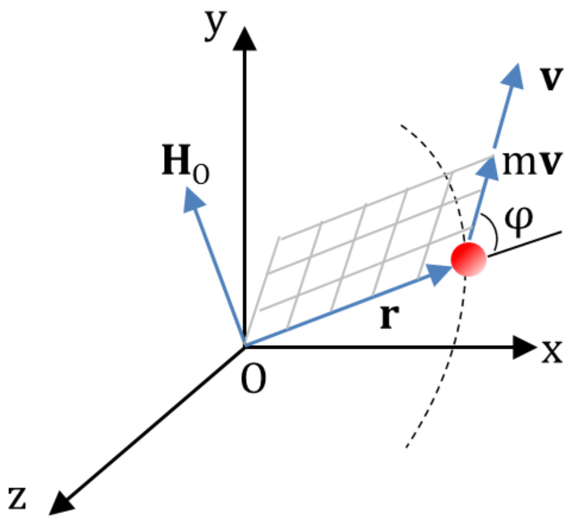


Figure 30 The definition of the angular momentum for a particle moving along a trajectory

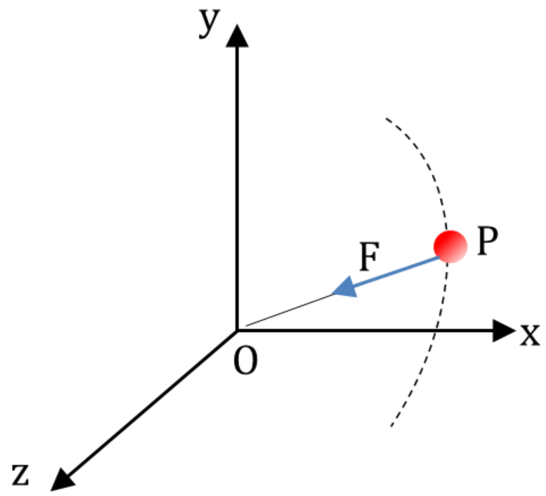


Figure 31 Particle moving under the influence of a central force

$\boldsymbol{H}_O$ is perpendicular to both $\boldsymbol{r}$ and $\boldsymbol{v}$ and is of magnitude

$$H_O = rmvsin\varphi = rmv_\theta = rm(r\dot{\theta}) = mr^2\dot{\theta}$$

The first important property of the angular moment is that it's time derivative equals the sum of moments

$$\dot{\boldsymbol{H}}_0 = \dot{\boldsymbol{r}} \times m\boldsymbol{v} + \boldsymbol{r} \times m\dot{\boldsymbol{v}} = \boldsymbol{v} \times m\boldsymbol{v} + \boldsymbol{r} \times m\boldsymbol{a} = \boldsymbol{r} \times \sum \boldsymbol{F} = \sum \boldsymbol{M} \qquad 28.$$

The second important property of the angular **momentum** is that it is constant if a particle is moving under influence of a central force. This follows from

$$\sum \boldsymbol{M} = 0 = \dot{\boldsymbol{H}}_O \rightarrow \boldsymbol{H}_O = \boldsymbol{r} \times m\boldsymbol{v} \text{ is constant} \qquad 29.$$

Which in particular is true for a planet moving under the influence of gravity. We may now define the angular momentum per unit mass as

$$h = \frac{H_O}{m} = \frac{mr^2\dot{\theta}}{m} = r^2\dot{\theta}$$

4.1.1. Calculated example: Velocity of the earth

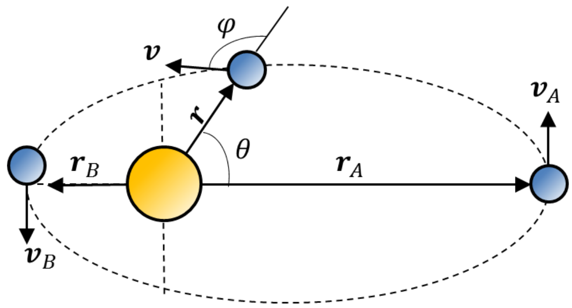
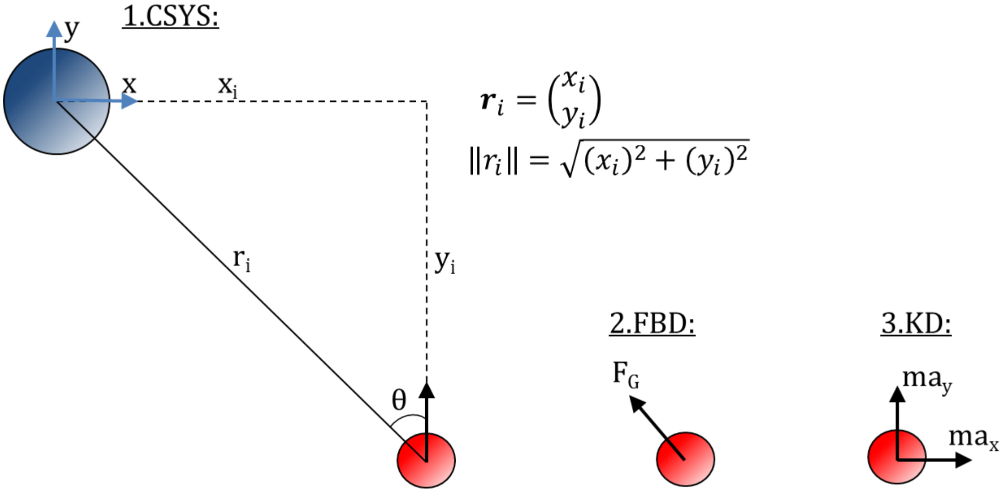


Figure 32 Conservation of the angular momentum for the earth orbiting the sun

Problem: If the lowest linear velocity of the earth orbiting the sun is known (in pos. A), determine the highest linear velocity (in pos. B).
Solution: Under influence of gravity, which is a central force, the angular momentum $H_O = rmvsin\varphi$ is conserved. For the two positions A and B, where **r** and **v** are perpendicular, it follows for $\varphi = 90$ deg that $\sin\varphi = 1$, so

$$r_A m v_A = r_B m v_B \rightarrow v_A = \frac{r_B}{r_A} v_B$$

4.1.2. Calculated example: Asteroid EoMs in castesian coordinates



$$\boldsymbol{r}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

$$\|r_i\| = \sqrt{(x_i)^2 + (y_i)^2}$$

Figure 33 The required diagrams for analysis of an asteroid

Problem: derive the equation of motion for an object with mass *m* in plane orbit around a larger object with mass *M*, for which m<<M. This could be the earth and an asteroid coming towards us, while we attempt to figure out if it will hit or pass us.

Solution: we mount a Cartesian CSYS in the CoGs. of *M*. The position vector of *m* will now be

$$\boldsymbol{r}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \text{ with } \|\boldsymbol{r}\| = \sqrt{{x_i}^2 + {y_i}^2}.$$

The gravity between *m* and *M* is given by

$$F_G = G\frac{mM}{\|r\|^2} \qquad \text{(Newton's law of gravity)}$$

Having drawn a FBD and a KD of *m*, we may do the dynamic equilibrium

$$\sum F_x = ma_x = -F_G \sin\theta = -F_G \frac{x_i}{\sqrt{{x_i}^2+{y_i}^2}}.$$

$$\sum F_y = ma_y = F_G \cos\theta = F_G \frac{-y_i}{\sqrt{{x_i}^2+{y_i}^2}}$$

Substituting $F_G = G\frac{mM}{\|r\|^2}$ into these equations, we get

$$\sum F_x = ma_x = -G\frac{mM}{\|r\|^2}\frac{x_i}{\sqrt{{x_i}^2+{y_i}^2}} \rightarrow \frac{d^2x}{dt^2} = -G\frac{Mx}{({x_i}^2+{y_i}^2)^{3/2}}$$

$$\sum F_x = ma_y = G\frac{mM}{\|r\|^2}\frac{-y_i}{\sqrt{{x_i}^2+{y_i}^2}} \rightarrow \frac{d^2y}{dt^2} = -G\frac{My}{({x_i}^2+{y_i}^2)^{3/2}}$$

or with $\boldsymbol{r}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$, we get $\frac{d^2}{dt^2}\boldsymbol{r} = -\frac{GM}{(\|r\|)^3}\boldsymbol{r}$

We have discovered a 2nd order differential equation – and these are among the finest of all things. Now we just need to come up with some way of solving it. We can do so using numerical integration like in the previous chapter. The code is reproduced on the following page.

```
for i=1:n+1
       t(i)=tlim/n*(i-1);
end
dt=t(2)-t(1);

r(1,:)=[xi,yi]; v(1,:)=[0,vi];
for i=2:n+1;
    a(i,:)=-G*M/(r(i-1,1)^2+r(i-1,2)^2)^1.5*r(i-1,:);
    v(i,:)=v(i-1,:)+a(i,:)*dt;
    r(i,:)=r(i-1,:)+v(i,:)*dt;
    H(i)=norm(cross([r(i,:) 0],m*[v(i,:) 0]));
    if (r(i,1)^2+r(i,2)^2)^0.5<=(r1+r2);
    t_imp=i; r(i,:)=r(t_imp,:); dt=0; end
end
```

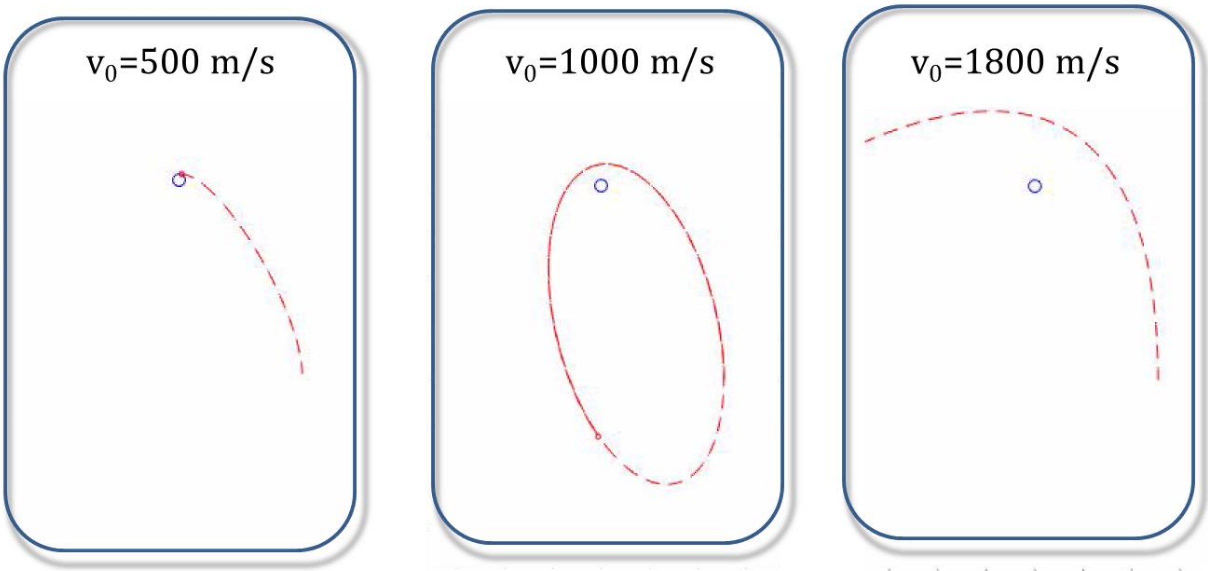


Figure 34 Examples of types of orbits

4.1.3. Calculated example: Doing a small solar system with the Kraken

```
PlotCoGTrcs="on"; PlotBdbs= "off"; Bdy2BdyCts="on";
Bdb="off";        CstG="off";      VarG="on";
SolverType="Fix";
k_b2b=10^15;
PltLim=[-10000000,10000000,-10000000,10000000]*2.5;
nt=10000;
tlim=100000;
PltFct=1;
%Body 1
IniCnd=[0,0,0,0,0,0];
GenCrcBdy(IniCnd,5.9*10^24,5.9*10^30,1100000);
%Body 2
IniCnd=[5*1100000,0,0,0,10000,0];
GenCrcBdy(IniCnd,10^18,10^18,5*110000);
%Body 3
IniCnd=[7*1100000,0,0,0,7500,0];
GenCrcBdy(IniCnd,10^18,10^18,5*110000);
%Body 4
IniCnd=[9*1100000,0,0,0,6000,0];
GenCrcBdy(IniCnd,10^18,10^18,5*110000);
%Body 5
IniCnd=[12*1100000,0,0,0,5000,0];
GenCrcBdy(IniCnd,10^18,10^18,5*110000);
```

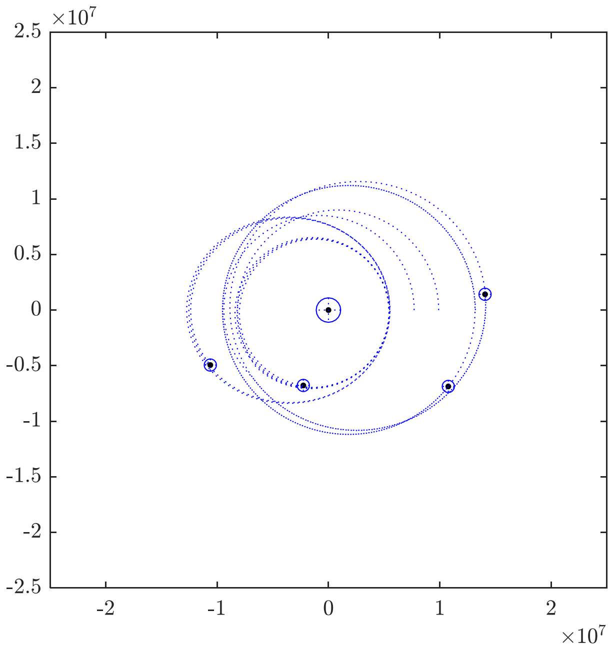


Figure 35 Result visualization

4.2. The rocket equation (First brief 'dark arts'-like topic)

For a particle system changing mass (for example a rocket), one would believe that the right form of equation for a would be

$$\sum F = \frac{d}{dt}(m(t)v(t)) = m(t)\frac{dv}{dt} + v(t)\frac{dm}{dt} \qquad 30.$$

which is obtained by differentiation of the linear momentum using the chain rule. This form is not mass invariant in the sense that the resultant force will be different for different reference frames. The correct form is

$$\sum \boldsymbol{F} + \boldsymbol{u}\frac{dm}{dt} = m\frac{dv}{dt} \qquad 31.$$

In which **u** is velocity of the escaping mass relative to the moving rocket. This term is often taken into account when calculating the thrust force produced by a rocket, and if this is the case Newton's 2nd law can be applied on it's conventional form. We're not going to make a bigger deal out this than we have to, but it is eventually crucial to remember the result above if you even encounter a system exerting mass, like for example a rocket.

4.3. Newton's solution to the planetary orbits (Second brief 'dark arts'-like topic)

Finally, we will briefly take a look at Newton's own solution to the planetary orbits – just so we all can say that we have seen this mighty mechanics result.

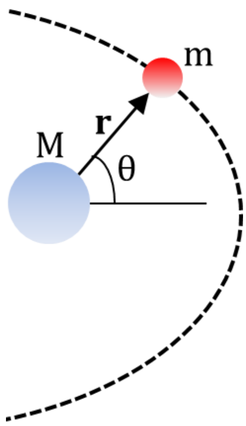


Figure 36 Planetary orbit in polar coordinates

In polar coordinates, using equilibrium based on Newton's 2nd law (and quite a bit of dark arts mathematics), gives us the differential equation:

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{mh^2u^2}$$

$$\mathrm{u} = \frac{1}{r}, F_G = G\frac{mM}{\|r\|^2}, \mathrm{h} = \dot{\theta}r^2$$

which is another magnificent differential equation of 2nd order, which in polar coordinates has the solution

$$u = \frac{1}{r} = \frac{GM}{h^2}(1 + \varepsilon cos\theta)$$

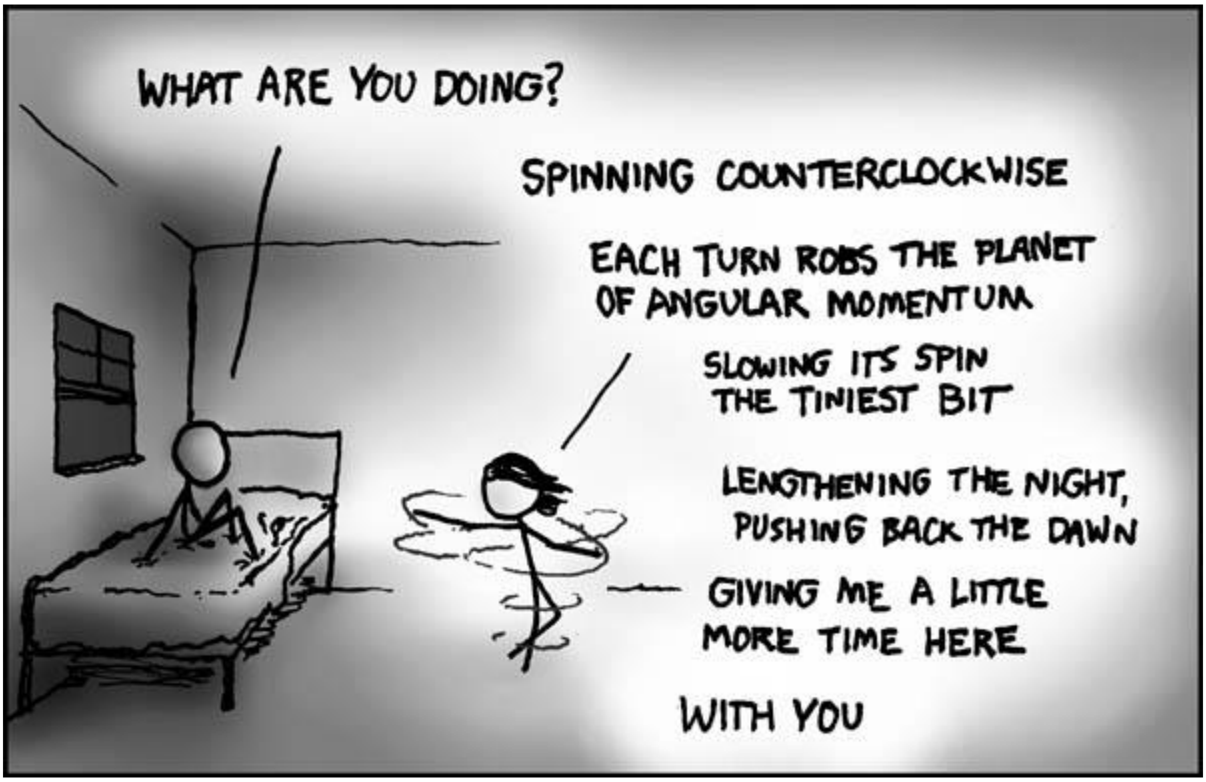
with eccentricity $\varepsilon = \frac{Ch^2}{GM}$ and C as an integration constant dependent on the initial conditions.

For $C > \frac{GM}{h^2}$: orbit is hyperbolic

For $\mathrm{C} = \frac{GM}{h^2}$: orbit is a parabola

For $C < \frac{GM}{h^2}$: orbit is elliptical

Congratulations – you have survived chapter 4. This is one of the rare occasions where mechanics becomes romantic



5. Mechanical vibrations

In this chapter, we will apply Newton's 2nd law to derive the equation(s) of motion for vibrating systems constituted by masses, springs and dampers. We will see, that the dynamics equilibrium can be derived to be on the form of 2nd order linear differential equations in time. This is really good news, since we from maths know exactly how to solve those to obtain the equation of motion on analytical harmonic form for the four common cases of vibrations free-undamped, free-damped, forced-undamped and forced damped. Finally, it will be demonstrated how to derive the dynamic equilibrium for systems with multiple degrees of freedom applying linear algebra.

The general dynamic equilibrium for a mass-spring-damper system subjected to a harmonic force is the most central equation in the present chapter. We will later see, that this can be extended to describe the vibrations of systems with multiple degrees of freedom using vectors and matrices.

5.1. Single DOF-vibrations

Vibrating systems are characterized by having a mass, a stiffness and a damping as properties. While mass and stiffness are usually rather easily determined, the system damping though often drawn as a damper, is very often caused by complex phenomena. Actually, literally anything dissipating energy introduces damping. Though more complex mathematical models for complex damping inducing phenomena (for example aeroelastic effects, material hysteresis etc.) to some extend have been developed, the parameters required for those are usually difficult to obtain. Vibration models based on velocity proportional – or viscous damping are therefore by far the most common description of vibrating systems due to their simplicity. However, the most challenging part of this type of analysis often remains actually to determine a damping to apply.
In general, we will differ between *free*- and *forced* vibrations. Free vibrations in general refers to the vibrating motion of a body due to a shock excitation or when released after having been subjected to a forced displacement, i.e. harmonic force excitations are absent. The induced oscillating motions will occur at rates defined by the eigenfrequencies of the system. Contrary to free-vibration, forced vibrations occur with the same frequency as the applied harmonic excitation force while also triggering the free-vibration response.
In general, when referring to degrees of freedom for vibrating systems, we basically mean the minimum number of independent coordinates required to describe the motion of the system. For 1D translating vibrations, one coordinate is obviously required for describing the position of each mass.
In case you have forgotten, you have already been cheated to do the analysis of a vibrating (or oscillating system), namely the pendulum considered in section 3.1.4. So though the content in the current section may seem mighty theoretical, you actually already have started out working with this.

5.1.1. The general equation of motion for damped-forced vibrations

Initially, a mass-spring-damper system subjected to a harmonic force excitation $F(t) = F_m sin(\omega_n t)$ will be considered, see Figure 37. This corresponds to the most complex of the four basic cases (the forced-damped case) to be considered. We will apply Newton's 2nd law to derive the dynamic equilibrium for the system, and the derived equation can afterwards to simplified in order to obtain the equation of motion for the cases to be considered. Assuming the damping viscous, we have

$$F_D = cv$$

Based on the FBD and KD in Figure 37, the dynamic equilibrium is given by

$$\sum F = ma = -cv - k(x+\delta) + mg + F(t) \qquad 32.$$

In this equation, we let δ denote the static spring deflection due to gravity and x denote the dynamic spring deflection due to motion of the mass. However, realizing that $k\delta = mg$, this can be rewritten to

$$ma + cv + kx = F(t) \rightarrow m\ddot{x} + c\dot{x} + kx = F(t) \qquad 33.$$

This means that gravitational effects have been reduced out of the dynamic equilibrium and therefore will not effect the motion amplitude nor frequency, but only provides a constant downwards offset. Therefore, gravity must for translational vibrations not be accounted for, and if the total dynamic spring elongation is required, it can be added to x as a constant term.

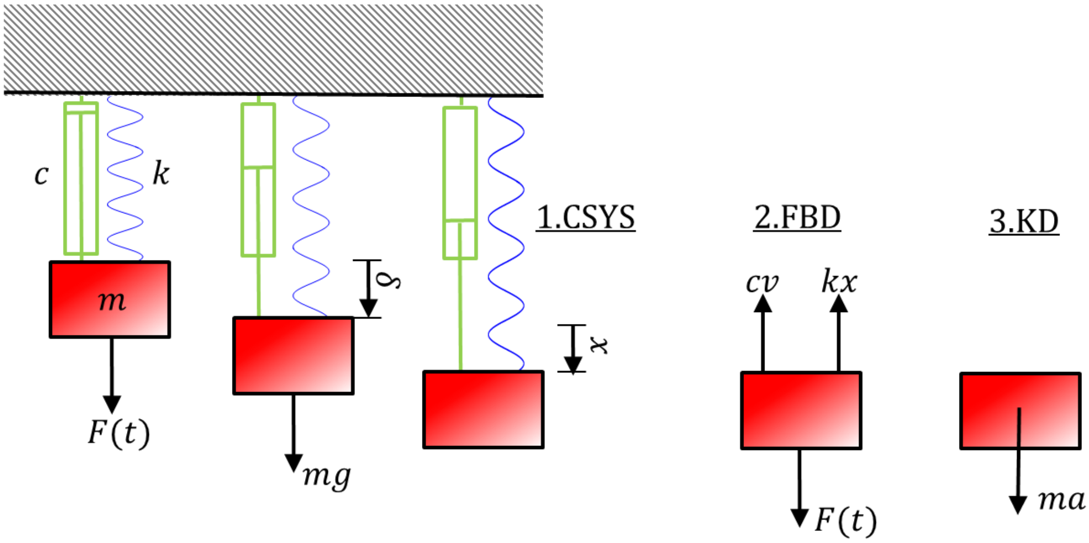


Figure 37 Mass-spring-damper system subjected to a harmonic excitation force

It turns out to be highly convenient to develop a numerical framework to apply as benchmark for the analytical solution which will be developed in the following sections (if you ask me why, you have to have just a little bit of faith – you have no idea have many answers that question has). We will as usual apply a simple first order numerical integrator and implement this in Matlab. As a first step, we will rearrange equation 33 by isolating the acceleration

$$a = \frac{1}{m}(F(t) - cv - kx)$$

Solving this on a mesh grid in time, we for the *i'th* timestep have the Forward-Euler scheme

$$(v)_{i+1} = (v)_i + \Delta t(a)_i \qquad 34.$$
$$(x)_{i+1} = (x)_i + \Delta t(v)_i$$

in which the accuracy can be improved by using the Implicit-Euler trick and modifying the second equation to

$$(\boldsymbol{x})_{i+1} = (\boldsymbol{x})_i + \Delta t(\boldsymbol{v})_{i+1}$$

Implementing this in Matlab we have the following code, which is going to turn out as useful.

```
%Kill all
clc; close all; clear all;
%Define system parameters
m=5; k=1000; c=0;
Fm=150; omega_f=10;
%Set initial values
xi=0.1; vi=0;
%Define numerical parameters
n=1000; tlim=10;
%Setup time vector
for i=1:n+1; t(i)=tlim/n*(i-1); end
dt=t(2)-t(1);
%Initialize arrays and set initial conditions
x(n+1)=0; x(1)=xi; v(n+1)=0; v(1)=vi;
%Do time integration
for i=2:n+1;
    v(i)=v(i-1)+dt*1/m*(Fm*sin(omega_f*t(i-1))-c*v(i-1)-k*x(i-1));
    x(i)=x(i-1)+dt*v(i);
end
%Plot solution
figure; plot(t,x); xlabel('time [s]'); ylabel('x [m]')
%Do FFT
dt=n/(tlim);
xdft = fft(x);
xdft = xdft(1:length(x)/2+1);
freq = linspace(0,dt/2,length(x)/2+1);
figure; plot(freq,abs(xdft),'b');
axis([0 5 0 50])
```

5.1.2. Case 1: Free undamped vibrations

In the case where no damping and no external excitation force is added to the system in Figure 37, the dynamic equilibrium in equation 33 reduces to

$$\sum F = m\ddot{x} = -\mathrm{kx} \rightarrow \ddot{x} + \frac{k}{m}x = 0$$

The natural circular frequency of the motion is recognized to be

$$\omega_n = \sqrt{\frac{k}{m}} \quad \left[\frac{\mathrm{rad}}{\mathrm{s}}\right]$$

The natural frequency in Hz (s^{-1}) can as usual be obtained by $f_n = \omega_n/(2\pi)$ and the period of the motion is $T_n = 1/f_n$. The dynamic equilibrium is on the form of a 2nd order linear homogenous differential equation with constant coefficients. From maths we now that the solution is given by

$$x(t) = C_1\cos(\omega_n t) + C_2\sin(\omega_n t)$$

The integration constants can be determined based on the initial velocity $\dot{x}(0) = v_0$ and initial position $x(0) = x_0$

$$x(0) = C_1\cos(\omega_n \cdot 0) = x_0 \rightarrow C_1 = x_0$$

$$\dot{x}(0) = \omega_n C_2\cos(\omega_n \cdot 0) = v_0 \rightarrow C_2 = \frac{v_0}{\omega_n}$$

From section 3.1.4, we know that the equation of motion can be rewritten to contain only a single harmonic term on the form $x(t) = X_m sin(\omega_n t + \varphi)$ with amplitude

$$X_m = \sqrt{{C_1}^2 + {C_2}^2} = \sqrt{{x_0}^2 + \left(\frac{v_0}{\omega_n}\right)^2}$$

The physical interpretation of this is that the system will undergo a harmonic motion, which corresponds with the equation of motion. However, in the absence of force, a motion can only be initiated by either moving the system away from the equilibrium position x_0 and releasing it or applying an initial velocity v_0.

While this model is more than useful and very commonly applied in mechanical vibration design, it contains a basic physical fault. No energy is ever dissipated from the system, and as a consequence the motion will once initiated continue forever. No real-life systems behave like that and the obtained model actually violates the basic laws of thermodynamics. As a consequence, free-damped vibrations will be considered in the following section. The basic mathematical description is still, in particular the system eigenfrequency, is of major importance, since a forced excitation at this frequency in poorly damped systems will lead to large amplitudes and possibly catastrophic failure. This phenomenon is often referred to as resonance, and we will get back to that in the following.

With the following input parameters, a motion can be initiated either applying an initial velocity, see Figure 38 and Figure 39, or an initial position, see Figure 40 and Figure 41.

```
m=5; k=1000; c=0;
Fm=0; omega_f=0;
```

The motion form can be observed to be harmonic as expected. The natural frequency can be calculated as

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000N/m}{5kg}} = 14.14\frac{rad}{s} \rightarrow f_n = \frac{\omega_n}{2\pi} = 2.25\ Hz$$

which corresponds to the peaks shown in Figure 39 and Figure 41. The amplitudes can be calculated by $X_m = \sqrt{{x_0}^2 + \left(\frac{v_0}{\omega_n}\right)^2}$ and correspond to the shown results.

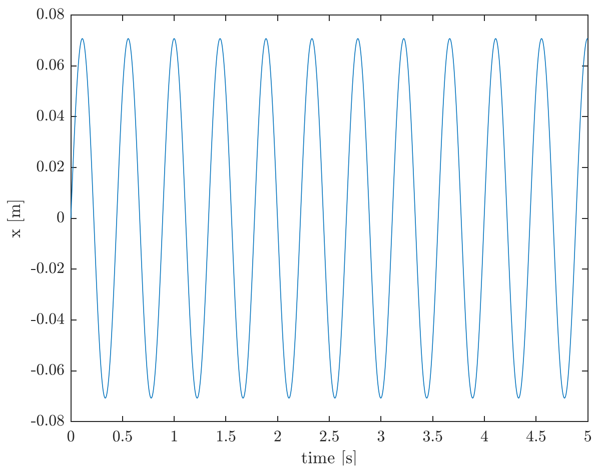


Figure 38 Motion response for initial conditions `xi=0.; vi=1;`

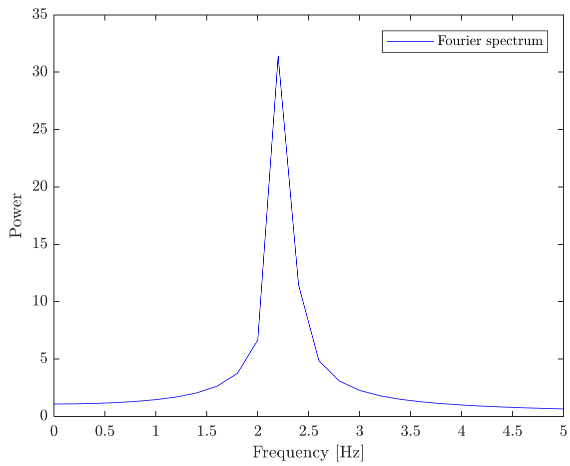


Figure 39 Frequency spectrum of the motion response in Figure 38

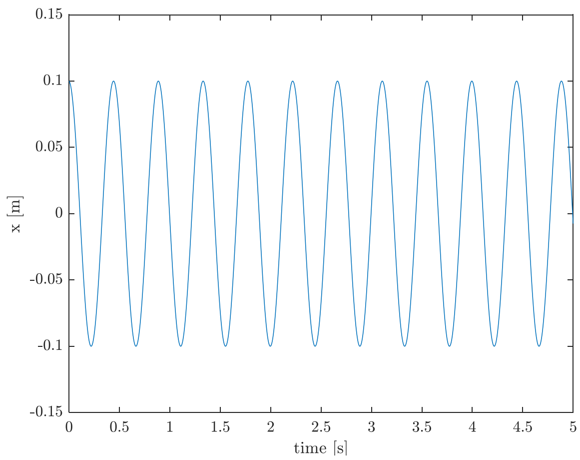


Figure 40 Motion response for initial conditions `xi=0.; vi=1;`

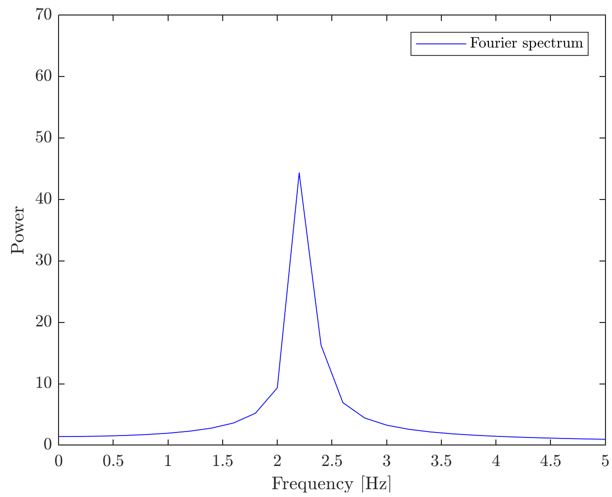


Figure 41 Frequency spectrum of the motion response shown in Figure 40

5.1.3. Case 2: Free damped vibrations

If damping is now added to the system considered in the previous section, the dynamic equilibrium in equation 33 is on the form

$$\ddot{x} + c\dot{x} + kx = 0$$

We will now define the damped cyclic eigenfrequency

$$\omega_d = \omega_n\sqrt{1-\left(\frac{c}{c_c}\right)^2}$$

in which the critical damping is given by

$$c_c = 2m\omega_n = 2\mathrm{m}\sqrt{\frac{k}{m}}$$

Damping can be observed to lower the circular frequencies and thereby the natural frequencies. Again, we have encountered a 2nd order linear homogenous differential equation with constant coefficients. From maths we know that we may define

$$\lambda_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

The governing equation has the following solutions dependant on how large the damping is:

Heavy damping, $c>c_c$ $\quad x = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}$

Critical damping, $c=c_c$ $\quad x = (C_1 + C_2t)e^{-\frac{c}{2m}t}$

Light damping $c<c_c$ $\quad x = e^{-\frac{c}{2m}t}\left(C_1\cos(\omega_d t) + C_2\sin(\omega_d t)\right)$

We note that the amplitudes are not constant but will decrease in time. For the heavy damping case, an oscillating motion will never make it through an entire vibration cycle. For practical purposes, the lightly damped case is usually considered the most important of the three cases.

With the following system input

```
m=5; k=1000; c=5;
Fm=0; omega_f=0;
```

The critical damping is given by

$$c_c = 2m\omega_n = 2(5\,kg)\left(14.14\,\frac{rad}{s}\right) = 141.4\,\frac{kg}{s}$$

Hence, the system is with an actual damping of 5kg/s lighly damped. This corresponds to the result shown in Figure 42. The eigenfrequency found in the frequency spectrum in Figure 43 can be observed to only be effected very little. The actual damped eigenfrequency is given by

$$\omega_d = \omega_n\sqrt{1-\left(\frac{c}{c_c}\right)^2} = \left(14.14\,\frac{rad}{s}\right)\sqrt{1-\left(\frac{5}{141.4}\right)^2} = 14.13\,\frac{rad}{s}$$

which is indeed a very limited influence.

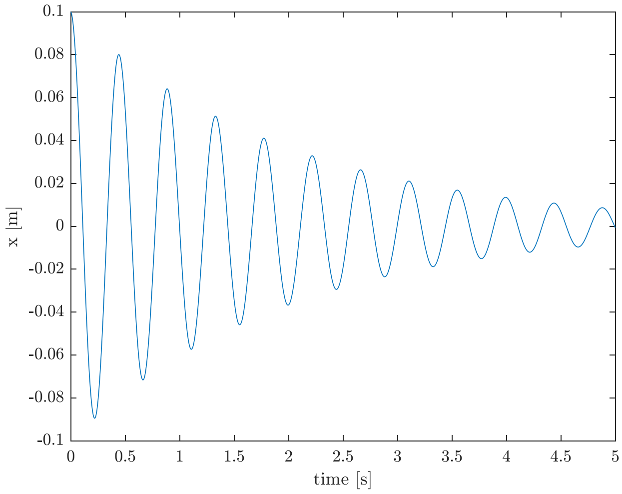


Figure 42 Lightly damped vibrations with initial conditions `xi=0.1; vi=0;`

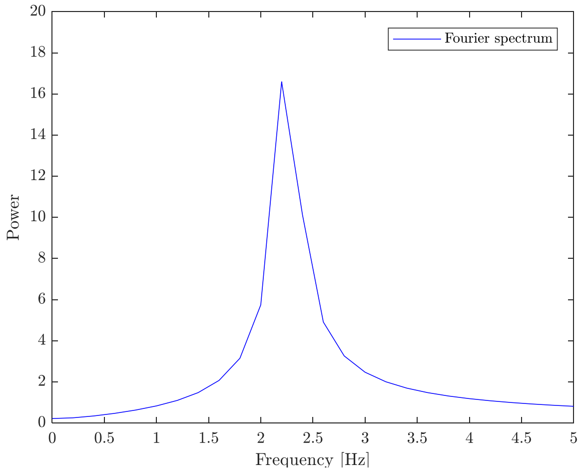


Figure 43 Frequency sprectrum

5.1.4. Case 3: Forced undamped vibrations

Now the case where a harmonic force is added to an undamped system will be considered. The dynamic equilibrium in equation 33 becomes

$$m\ddot{x} + kx = P_m sin(\omega_f t)$$

If a moving base is added instead of a driving force, the dynamic equilibrium is

$$m\ddot{x} + kx = k\delta_m sin(\omega_f t)$$

The motion response will be a sum of complementary (free) and particular (forced steady state) vibrations

$$x(t) = [C_1\cos(\omega_n t) + C_2\sin(\omega_n t)] + X_m sin(\omega_f t)$$

In all real life systems, energy dissipation (i.e. damping) will cause the free vibrations to vanish leaving only the steady-state (particular) term $x(t) = X_m sin(\omega_f t)$ with

$$X_m = \frac{\frac{P_m}{k}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}, \delta_m = \frac{P_m}{k}$$

For forced vibrations, it in general applies that:
For $\omega_f < \omega_n$: motion and force are in-phase
For $\omega_f > \omega_n$: motion and force are out-of-phase

5.1.5. Case 4: Forced damped vibrations

Finally, if a harmonic force excitation is added, the dynamic equilibrium becomes

$$m\ddot{x} + c\dot{x} + kx = P_m sin(\omega_f t)$$

This equation has the steady-state response $x(t) = X_m sin(\omega_f t)$ with amplitude

$$X_m = \frac{\frac{P_m}{k}}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2+\left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}}$$ [13]

If the spring is mounted on a base in harmonic motion, $\delta(t) = \delta_m \sin(\omega_f t)$, we may set $\delta_m = \frac{P_m}{k}$ in the formulaes above

With the following system input

```
m=5; k=1000; c=5;
Fm=150; omega_f=10;
```

the system response if shown in Figure 44 for the force input shown in Figure 45. The motion can be observed to be irregular for a few vibration cycles, which in a physical sense corresponds to the presence of free vibrations. This is usually called *the transient state*. As free vibrations are damped out, the motion becomes regular and constitute a harmonic response. This is called *the steady state.* Since the frequency of the excitation force is lower than the circular frequency, force and motion response is in phase.

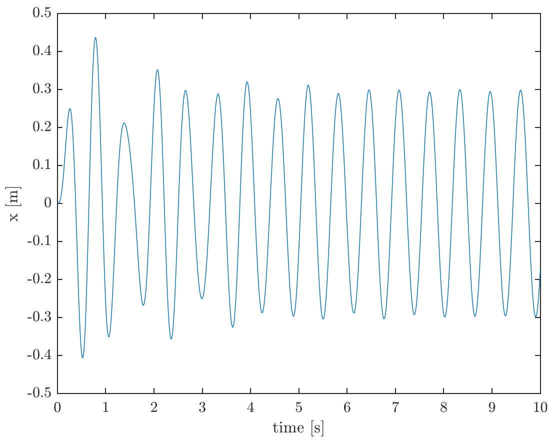


Figure 44 Example of forced vibration in-phase motion response

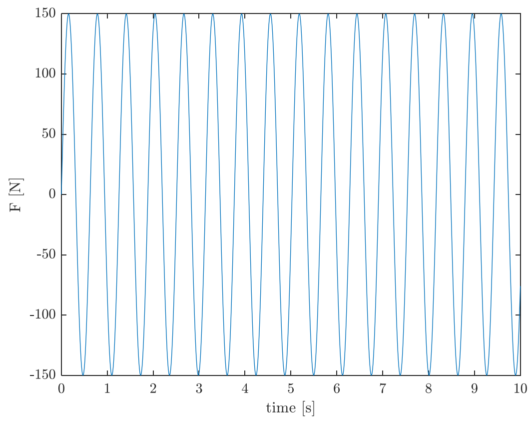


Figure 45 The corresponding force excitation

If the frequency of the forced excitation is increased above the circular frequency with the input

```
m=5; k=1000; c=25;
Fm=150; omega_f=20;
```

the motion response shown in Figure 46 is obtained. Once steady state is reached, the motion can be observed to be out-of-phase with the force response shown in Figure 47. If an FFT of the motion response is conducted, the frequency content can now be observed to contain both the natural frequency and the forced excitation frequency, see Figure 48.

[13] this expression is widely accepted as the most horrible equation in mechanical engineering – All mechanics professors had a meeting and decided that it is like that.

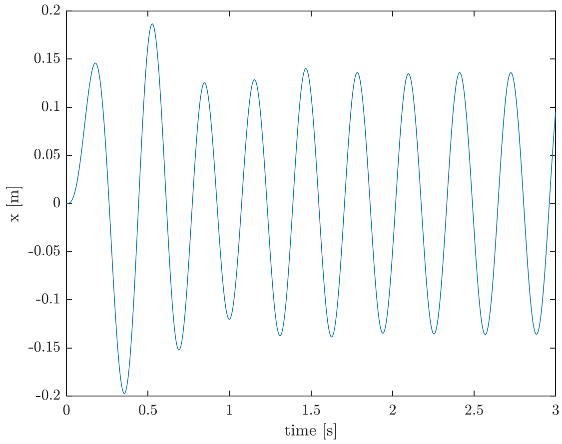


Figure 46 Example of forced vibration in-phase motion response

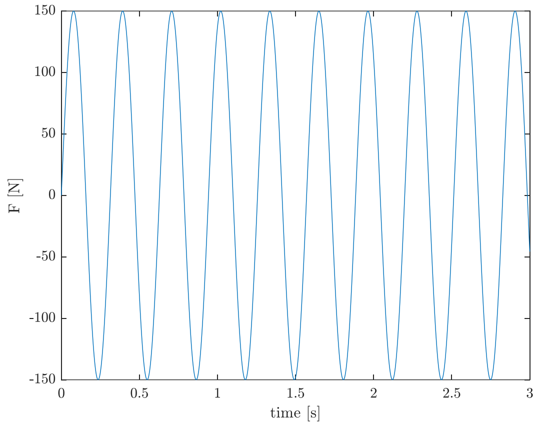


Figure 47 The corresponding force excitation

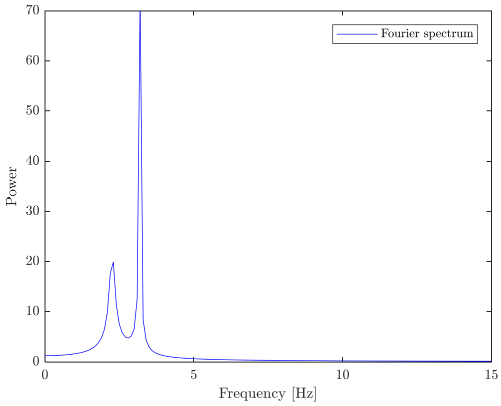


Figure 48 Example of frequency spectrum from

Finally, if the forced excitation is set equal to the natural frequency, meaning that $\omega_f = \omega_n$, for an undamped system (c=0), resonance is triggered. For an undamped system, energy will be accumulated in the system and the amplitudes will in theory become infinitely large. This would in real life systems lead to catastrophic failure, see Figure 49. If an excitation frequency close to the circular frequency is applied, a weird looking oscillating phenomenon known as beating may be triggered, see Figure 50.

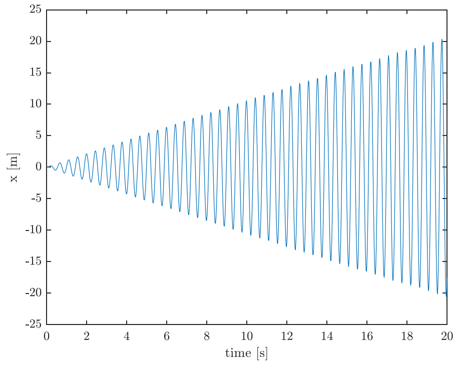


Figure 49 Motion response for ω_f =14.14 rad/s (ω_n) for undamped system, resonance

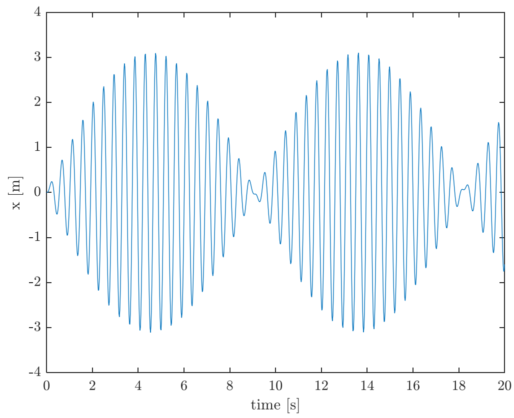


Figure 50 Motion response for ω_f =13.5 rad/s (close to ω_n), beating

5.1.6. Calculated example: Single DOF mass-spring-damper system

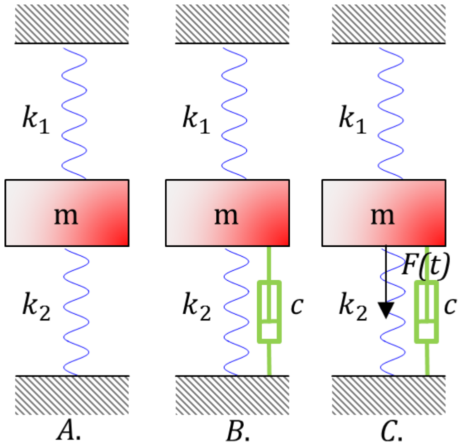


Figure 51 Mass-spring-damper system

Problem: In the system shown in Figure 51, a mass m=5 kg is supported by two springs with stiffness k_1=11000 N/m and k_2=k_3=7800 N/m.

1. For the undamped system shown in Figure A, calculate the natural frequency

If the undamped system shown in Figure A is stroke with a hammer inducing a downwards velocity of 0.7 m/s, calculate

2. The amplitude of the motion response
3. The amplitude of the acceleration response

In order to minimize the motion amplitude, a damper is added to the system, see Figure B.

4. Calculate the minimum damping coefficient for which the system will be in a non-vibratory mode

However, you decide only to apply half of the damping calculated in 4).

5. Determine the damped eigenfrequency of this system

A harmonic force of amplitude F_m=270 N and circular frequency ω_f=20 rad/s is applied to the mass, see figure C.

6. Determine the amplitude of the steady state forced-damped motion response.

Solution:

1. Eigen frequency: In order to obtain the system stiffness to be applied, we observe that both springs must experience deformations of equal magnitude since they are mounted on each side of the mass, which is the definition of springs-in-parallel (though they might seem mounted in series). I.e. $k = k_1 + k_2$

Natural circular (or cyclic) frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k_1+k_2}{m}} = \sqrt{\frac{(11000+7800)\frac{N}{m}}{5\,kg}} = 61.3\,\frac{rad}{s}$$

The natural or eigen frequency is now given by

$$f_n = \frac{\omega_n}{2\pi} = \frac{61.3\,\frac{rad}{s}}{2\pi\,rad} = 9.76\,s^{-1}$$

2. Amplitude of the motion response (v_0=0.7 $\frac{m}{s}$, x_0=0):

$$X_m = \sqrt{(x_0)^2 + \left(\frac{v_0}{\omega_n}\right)^2} = \frac{v_0}{\omega_n} = \frac{0.7\frac{m}{s}}{61.3\,\frac{rad}{s}} = 0.0114\,m$$

3. Amplitude of acceleration response:

$$x(t) = X_m \sin(\omega_n t) \rightarrow a(t) = \ddot{x}(t) = -(\omega_n)^2 X_m \sin(\omega_n t)$$

$$a_m = (\omega_n)^2 X_m = 0.0114\,m\left(61.3\,\frac{rad}{s}\right)^2 = 42.9\,\frac{m}{s^2}$$

4. Critical damping: The system is in non-vibratory mode if the damping is above the critical damping ($c>c_c$):

$$c_c = 2m\omega_n = 2 \cdot 5\text{kg} \cdot 61.3\,\frac{rad}{s} = 613\,\frac{\text{kg}}{\text{s}}$$

5. Damped eigenfrequency:

$$\omega_d = \omega_n\sqrt{1-\left(\frac{c}{c_c}\right)^2} = \omega_n\sqrt{1-\left(\frac{0.5c_c}{c_c}\right)^2} = \omega_n\sqrt{1-(0.5)^2}$$

$$61.3\,\frac{rad}{s}\sqrt{1-0.5^2} = 53.1\,\frac{rad}{s}$$

6. Amplitude of motion for forced response

$$X_m = \frac{\frac{P_m}{k}}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2+\left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}}$$

$$= \frac{\frac{270\,N}{(11000+7800)\frac{N}{m}}}{\sqrt{\left[1-\left(\frac{20}{61.3}\right)^2\right]^2+\left[2(0.5)\left(\frac{20}{61.3}\right)\right]^2}} = 0.0151m$$

(in-phase with force since $\omega_f < \omega_n$)

5.1.7. Forced excitation with increasing frequency: the linear chirp signal

Rotating machinery does not operate at a constant rotational speed, and will at least during run-up experience different frequencies until a given service speed is reached. We distinguish under-critical and over-critical operation dependent on if the rotational frequency is below or above the lowest eigenfrequency. For over-critical rotational speeds, an eigenfrequency must be passed during acceleration, and it is for poorly damped systems often crucial to pass these frequencies fast in order to avoid failure. In order to account for this type of mechanical behavior, a mathematical description of a harmonic excitation is required. A harmonic motion with variable frequency can be described by a chirp signal on the form

$$x(t) = \sin(2\pi \int f(t)dt) \qquad 35.$$

A linear chirp signal is given by $f(t) = f_0 + kt$. The chirp signal is now given by

$$x(t) = \sin\left(2\pi \int_0^t f(t')dt'\right) = \sin\left(2\pi \int_0^t f_0 + kt'\,dt'\right) = \sin\left(2\pi\left(f_0+\frac{k}{2}t\right)t\right) \qquad 36.$$

An example of a linear chirp signal is shown in Figure 52.

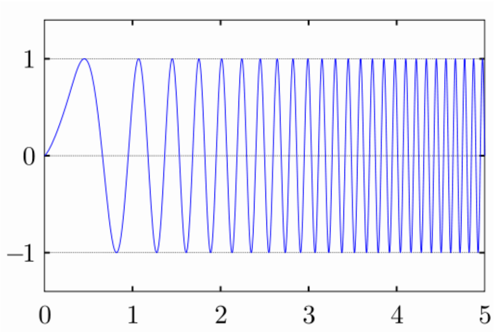


Figure 52 Example of a linear Chirp signal (foto from Wikimedia commons)

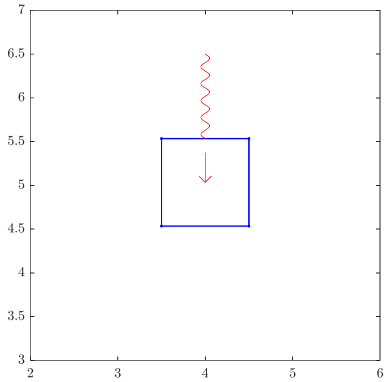


Figure 53 Kraken mass-spring-damper system subjected to a forced excitation with increasing frequency

In order to demonstrate how chirp excitations work, we will this time take the easy way out and use the Kraken, since she was taught chirping during a mechanical master project in 2019. We can define a model as shown in Figure 53 using the following input file

```
Mass-spring-damper system, kraken example model
%Mass-spring-damper system, kraken example model
%NHS,HSRW, 11.08.2020
%System parameters
m=5; k=1000; c=5;
Fm=15; omega_f=0; phi=0;
%Time integration parameters
nt=2000; tlim=25;
%Generate box geometry
Pts_Geb1=[0,0;0,1;1,1;1,0];
TriLns_Geb1=[1,2,3; 1,3,4];
IniCnd=[4,5,0,0,0,0];
GenGebBdy(IniCnd,m,c,Pts_Geb1,TriLns_Geb1)
%Attach spring
Bdy2WldTrsSpr(200,2,1,1,[0.,0.5],[IniCnd(1),IniCnd(2)+1.5]);
%Do harmonic chirp force
DOF=2; Bdy_Num=1; k_chirp=0.5;
SetHrmonicFrc(Fm,omega_f,phi,Bdy_Num,DOF,k_chirp)
%Settungs
PltFct=2; PltLim=[2,6,3,7]; PltBdyRsp(1,2)
PlotCoGs= "off"; PlotCoGTrcs="off"; PlotBdbs= "off";
CstG="off"; VarG="off"; SolverType="Var";
```

The obtained results are shown in Figure 54 and Figure 55 showing the effect of a run-up. The amplitudes can clearly be shown to be increasing as the eigenfrequency is passed and to drop afterwards as the frequencies above the eigenfrequency.

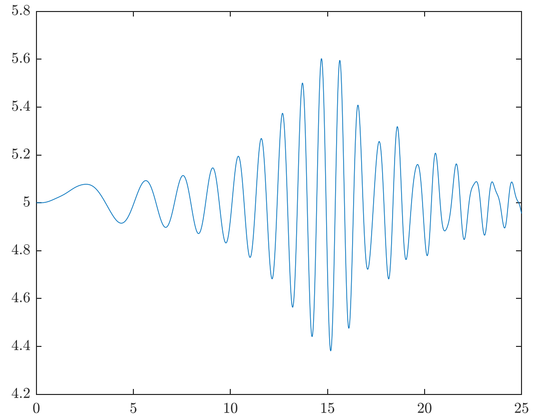


Figure 54 Vertical displacement as function of time

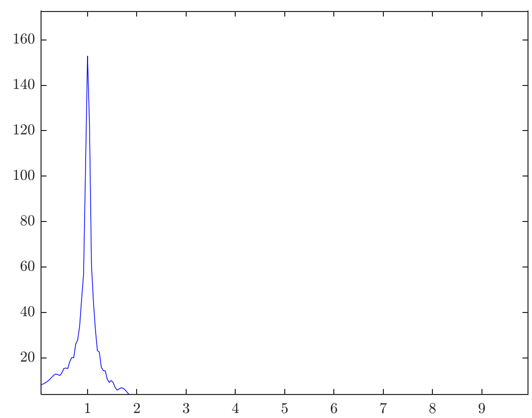


Figure 55 Frequency spectrum of the displacements shown in Figure 54

5.2. Multi DoF-vibrations

We will now consider how to conduct vibration analysis of systems consisting of more than one mass, i.e. having more than one degree of freedom, and we will see, that one equation of motion is required for each DoF. It is convenient to apply linear algebra and formulate mass, stiffness and damping on matrix form. Many equations of motion derived on basis of the dynamic equilibrium do have analytical solutions. However, these are beyond our current scope, and we will focus our efforts on deriving the dynamic equilibrium and on this basis calculating the eigenfrequencies of the considered systems. Contrary to the systems considered in the previous sections, which all were single DoF systems, a system with *n* DoF will have *n* eigenfrequencies to be determined as the solution to an eigen-value problem.
The theory in section 5.2.1 may seem abstract or complex, but kindly avoid panicking, since things should be a whole lot clearer when applying this for an actual calculated example in section 5.2.2.

5.2.1. Derivation of the dynamic equilibrium

Considering linear systems, with multiple DoF's, the dynamic equilibrium can in general be written on the form

$$[\mathbf{M}]\{\ddot{\boldsymbol{x}}\} + [\boldsymbol{C}]\{\dot{\boldsymbol{x}}\} + [\mathbf{K}]\{\boldsymbol{x}\} = \{\boldsymbol{F}\} \qquad 37.$$

$$\rightarrow [\mathbf{M}]\{\ddot{\boldsymbol{x}}\} = -[\boldsymbol{C}]\{\dot{\boldsymbol{x}}\} - [\mathbf{K}]\{\boldsymbol{x}\} + \{\boldsymbol{F}\}$$

$$\rightarrow \{\ddot{\boldsymbol{x}}\} = -[\mathbf{M}]^{-1}[\boldsymbol{C}]\{\dot{\boldsymbol{x}}\} - [\mathbf{M}]^{-1}[\mathbf{K}]\{\boldsymbol{x}\} + [\mathbf{M}]^{-1}\{\boldsymbol{F}\}$$

This corresponds exactly to the general single DoF equilibrium in equation 33, except the matrix formulation now allows us to handle multiple equations simultaneously. This equation can still be solved using implicit-Euler integration

$$\{\dot{\boldsymbol{x}}\}_{i+1} = \{\dot{\boldsymbol{x}}\}_i + \Delta t\{\ddot{\boldsymbol{x}}\}_i$$

$$\{\boldsymbol{x}\}_{i+1} = \{\boldsymbol{x}\}_i + \Delta t\{\dot{\boldsymbol{x}}\}_{i+1}$$

Now, focusing on determination of eigen-frequencies, the dynamic equilibrium of an undamped system is given by

$$[\mathbf{M}]\{\ddot{\boldsymbol{x}}\} + [\mathbf{K}]\{\boldsymbol{x}\} = 0 \qquad 38.$$

Assuming the solution harmonic (for all DOFs) we obtain

$$\{\boldsymbol{x}\} = \{\boldsymbol{x}\}_m \sin(\omega_n t + \varphi) \rightarrow \{\ddot{\boldsymbol{x}}\} = -(\omega_n)^2\{\boldsymbol{x}\}_m \sin(\omega_n t + \varphi)$$

Substituting this into the dynamics equilibrium, we obtain

$$[\mathbf{M}](-(\omega_n)^2\{\boldsymbol{x}\}_m \sin(\omega_n t + \varphi)) + [\mathbf{K}](\{\boldsymbol{x}\}_m \sin(\omega_n t + \varphi)) = 0$$

$$\rightarrow ([\mathbf{K}] - (\omega_n)^2[\mathbf{M}])(\{\boldsymbol{x}\}_m \sin(\omega_n t + \varphi)) = 0$$

This has to hold for all values of t, i.e.

$$([\mathbf{K}] - (\omega_n)^2[\mathbf{M}])\{\boldsymbol{x}\}_m = 0 \qquad 39.$$

We recognize this as an eigenvalue problem that can be solved for circular eigenfrequencies ω_n and eigenvectors $\{\boldsymbol{x}\}_m$. A two DOF system (like in the following calculated example) has maximum two eigenvalues. A continuous system has infinitely many eigenvalues.

It can be shown mathematically that the solution in ω_n to the equation can be obtained by setting the determinant of $[\mathbf{K}] - (\omega_n)^2[\mathbf{M}]$ equal to 0:

$$|[\mathbf{K}] - (\omega_n)^2[\mathbf{M}]| = \mathbf{0} \qquad 40.$$

This will lead to, what is usually referred to as the characteristic equation (or characteristic polynomial).

Math Recap: the determinant of a 2×2 matrix is given by

$$\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = ad - cb$$

For a 3×3 matrix, we have

$$\left|\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

This principle based on sub-determinants can be extended to matrices of arbitrary large size, but becomes computationally inefficient already for only moderately large systems.

And that's it. We are ready to run off to calculate eigenfrequencies of larger systems.

5.2.2. Calculated example: Two-DoF mass-spring-damper systems

Problem: Determine the dynamic equilibrium and the undamped eigenfrequencies for the system shown in Figure 56 for parametric input

Solution: We assume $x_A, x_B > 0$ and $x_A < x_B$

$$m_A\ddot{x}_A = -k_1x_A - c_1\dot{x}_A + k_2(x_B - x_A) + c_2(\dot{x}_B - \dot{x}_A) + F_A(t)$$
$$m_B\ddot{x}_B = -k_3x_B - c_3\dot{x}_B - k_2(x_B - x_A) - c_2(\dot{x}_B - \dot{x}_A) + F_B(t)$$

Rearranging these equations, we get

$$m_A\ddot{x}_A + (c_1 + c_2)\dot{x}_A - c_2\dot{x}_B + (k_1 + k_2)x_A - k_2x_B = F_A(t)$$
$$m_B\ddot{x}_B + (c_2 + c_3)\dot{x}_B - c_2\dot{x}_A + (k_2 + k_3)x_B - k_2x_A = F_B(t)$$

On matrix form, this gives us

$$\underbrace{\begin{bmatrix} m_A & 0 \\ 0 & m_B \end{bmatrix}}_{[\boldsymbol{M}]}\underbrace{\begin{Bmatrix} \ddot{x}_A \\ \ddot{x}_B \end{Bmatrix}}_{(\ddot{\boldsymbol{x}})} + \underbrace{\begin{bmatrix} (c_1 + c_2) & -c_2 \\ -c_2 & (c_2 + c_3) \end{bmatrix}}_{[\boldsymbol{C}]}\underbrace{\begin{Bmatrix} \dot{x}_A \\ \dot{x}_B \end{Bmatrix}}_{(\dot{\boldsymbol{x}})} + \underbrace{\begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix}}_{[\boldsymbol{K}]}\underbrace{\begin{Bmatrix} x_A \\ x_B \end{Bmatrix}}_{(\boldsymbol{x})} = \underbrace{\begin{Bmatrix} F_A(t) \\ F_B(t) \end{Bmatrix}}_{(\boldsymbol{F})}$$

This is the general dynamic equilibrium for the system and the terms representing mass, stiffness and damping are clearly recognized.

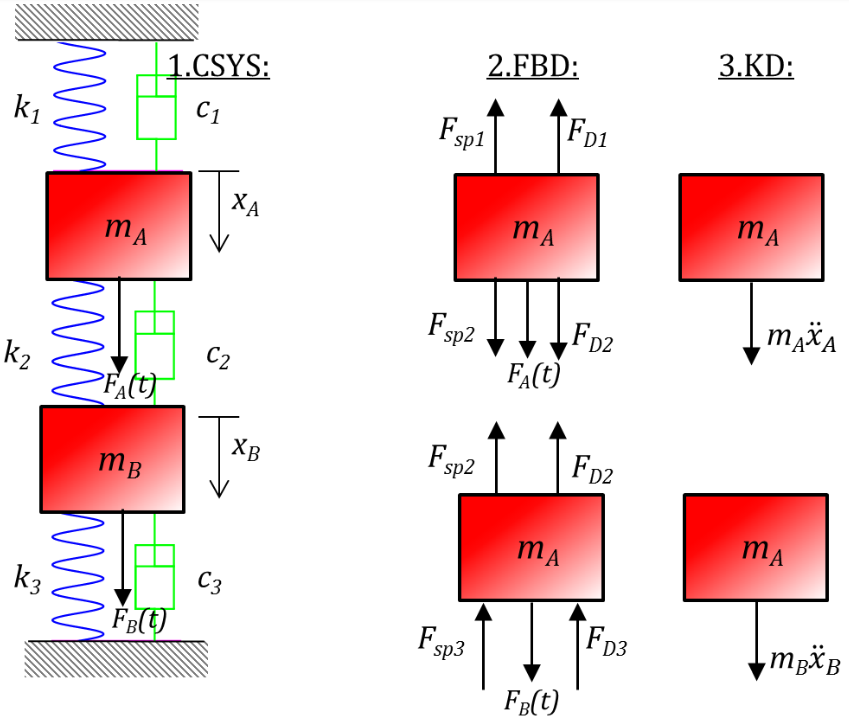


Figure 56 Two DoF mass-spring-damper system

For undamped free vibrations, the dynamic equilibrium reduces to

$$\begin{bmatrix} m_A & 0 \\ 0 & m_B \end{bmatrix} \begin{Bmatrix} \ddot{x}_A \\ \ddot{x}_B \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & (k_2+k_3) \end{bmatrix} \begin{Bmatrix} x_A \\ x_B \end{Bmatrix} = \mathbf{0}$$

In order to calculate the eigenfrequencies, we formulate this as eigenvalueproblem:

$$|[\mathbf{K}] - (\omega_n)^2[\mathbf{M}]| = \left\| \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & (k_2+k_3) \end{bmatrix} - (\omega_n)^2 \begin{bmatrix} m_A & 0 \\ 0 & m_B \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} (k_1+k_2) - m_A(\omega_n)^2 & -k_2 \\ -k_2 & (k_2+k_3) - m_B(\omega_n)^2 \end{bmatrix} \right\|$$

$$= \left((k_1+k_2) - m_A(\omega_n)^2\right)\left((k_2+k_3) - m_B(\omega_n)^2\right) - (-k_2)^2$$

$$(k_1+k_2)(k_2+k_3) - (k_1+k_2)m_B(\omega_n)^2 - (k_2+k_3)m_A(\omega_n)^2 + m_A m_B((\omega_n)^2)^2 - (k_2)^2$$

$$= m_A m_B((\omega_n)^2)^2 - \left((k_1+k_2)m_B + (k_2+k_3)m_A\right)(\omega_n)^2 + (k_1+k_2)(k_2+k_3) - (k_2)^2 = 0$$

We recognize this as a quadratic equation, $a((\omega_n)^2)^2 + b(\omega_n)^2 + c = 0$ with

$$a = m_A m_B$$

$$b = -\left((k_1+k_2)m_B + (k_2+k_3)m_A\right)$$

$$c = (k_1+k_2)(k_2+k_3) - (k_2)^2$$

Solving this using the code fragment shown below, we will for the specified system parameters get $\omega_1 = 31.6$ rad/s and $\omega_2 = 57.0$ rad/s corresponding to natural frequencies $f_1 = 5.0$ Hz and $f_2 = 9.1$ Hz

```
clc; close all; clear all
%Input system parameters
k1=2000; k2=3000; k3=4000; mA=2; mB=4;
%Do analytical solution to eig-problem
a=mA*mB;
b=-((k1+k2)*mB+(k2+k3)*mA);
c=(k1+k2)*(k2+k3)-k2^2;
omega1=sqrt((-b-sqrt(b^2-4*a*c))/(2*a))
omega2=sqrt((-b+sqrt(b^2-4*a*c))/(2*a))
```

We could have obtained exactly the same result using the numerical eigenvalue solver in Matlab by running the code shown below after the previous code.

```
%Do numerical solution
K=[k1+k2,-k2;-k2,k2+k3]; M=[mA,0;0,mB];
omega=eigs(K,M)
omega1_num=sqrt(omega(1))
omega2_num=sqrt(omega(2))
%Convert freq. from rad/s to Hz
fn1=omega1/(2*pi)
fn2=omega2/(2*pi)
```

The dynamic equilibrium could also be integrated numerically using the code shown below if the actual motion is required. The results are shown in Figure 57. However, the eigenfrequencies could also be extracted directly from any of these two responses by running an FFT on them, see Figure 58. The eigenfrequencies can be observed to correspond with the previously obtained results.

```
%Input system parameters
k1=2000; k2=3000; k3=4000; mA=2; mB=4;
K=[k1+k2,-k2;-k2,k2+k3]; M=[mA,0;0,mB];
%Initial conditions
x1=0.02; x2=0.04;
%Input numerical parameters
n=2000; tlim=10;
%Setup time vector
for i=1:n+1; t(i)=tlim/n*(i-1); end
dt=t(2)-t(1);
%Initialize arrays and set initial values
x(2,n+1)=0; v(2,n+1)=0; x(:,1)=[x1,x2];
%Do time integration
for i=2:n+1;
    v(:,i)=v(:,i-1)+dt*(inv(M)*(-K*x(:,i-1)));
    x(:,i)=x(:,i-1)+dt*v(:,i);
end
%Plot time response
figure; plot(t,x(1,:),t,x(2,:))
%Do FFT of x
dt=n/(tlim);                          %log rate
xdft = fft(x(2,:));
xdft = xdft(1:length(x(2,:))/2+1);
freq = linspace(0,dt/2,length(x(2,:))/2+1);
figure; plot(freq,abs(xdft),'b');
axis([0 15 0 30])
```

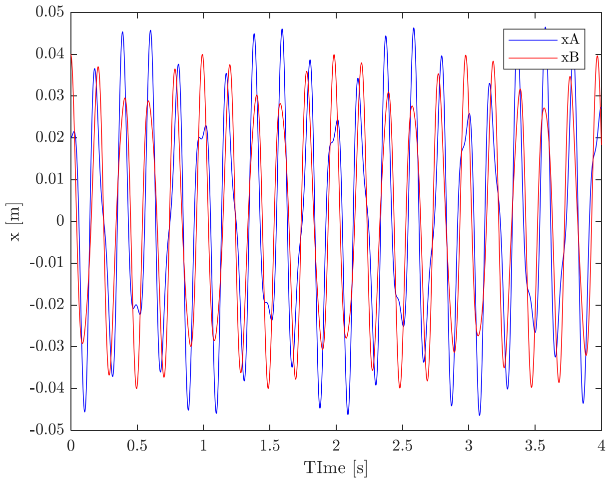


Figure 57 Motion response

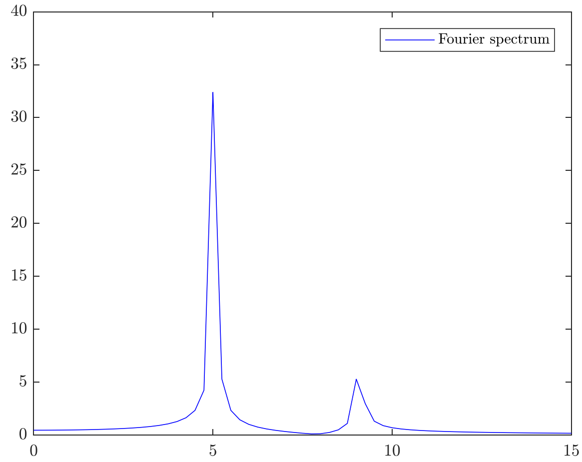


Figure 58 Frequency spectrum

5.3. Machine dynamics

The theory and techniques introduced in the present chapter can, if combined with stiffness measures from strength of materials and rigid body dynamics, be applied for analysis and eigenfrequency calculation of rotating machinery. For most high speed applications, it is however more accurate to base the analysis of measured accelerations rather than displacements or velocities. Furthermore, the frequency content from measurements may not only provide information about drive and eigenfrequencies, but also reveal faults in machinery, like defect bearings, imbalances and misalignments. An example of a setup for such experiments is shown in Figure 59 with obtained results shown in Figure 60-Figure 63.

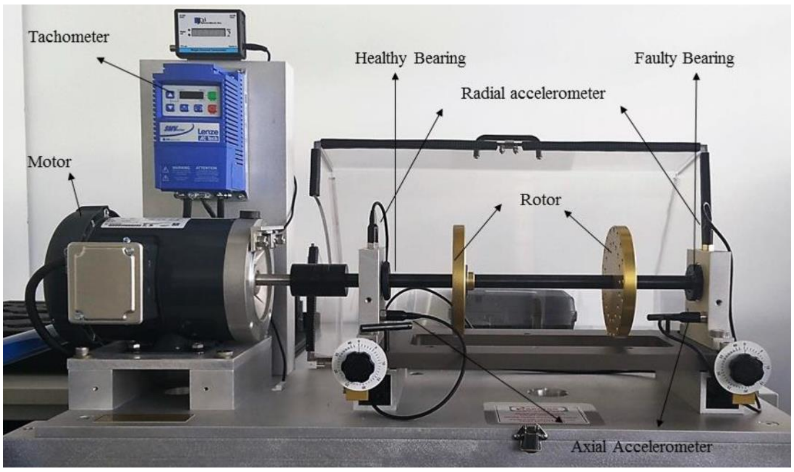


Figure 59 The fault simulator – a drive train for laboratory scale investigation of fault frequencies based on measured accelerations

This topic is unfortunately more than just a tick too broad for it to be dealt with in a one semester class on dynamics. That's why there's an entire elective, which only is about wrapping up rotor dynamics and rigid body vibrations.

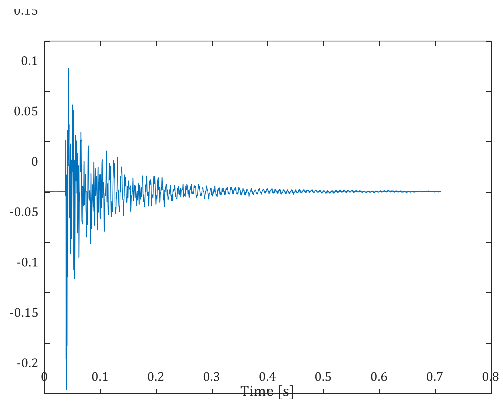


Figure 60 Accelerationslogged from a static impact test

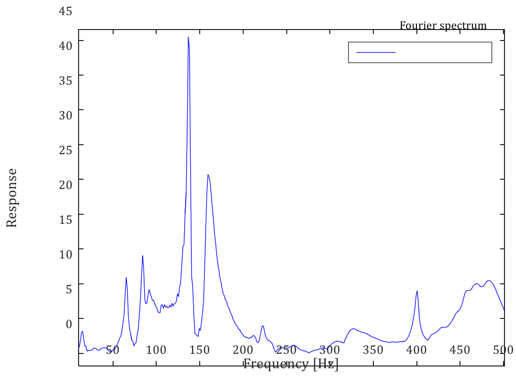


Figure 61 Frequency spectrum from impact test

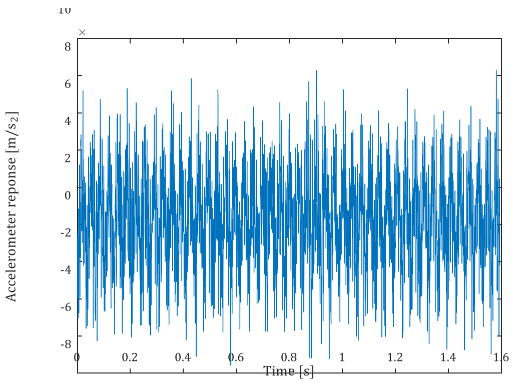


Figure 62 Accelerations logged from a single imbalanced rotor running at 30 Hz

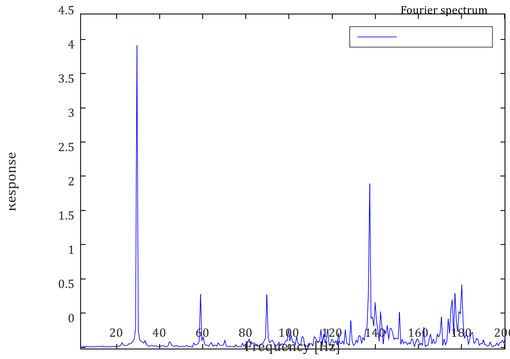


Figure 63 Frequency spectrum for imbalanced rotor

Congratulations – you have survived chapter 5.