Problems based introduction to machine dynamics

1. Introduction (what we're gonna do here)

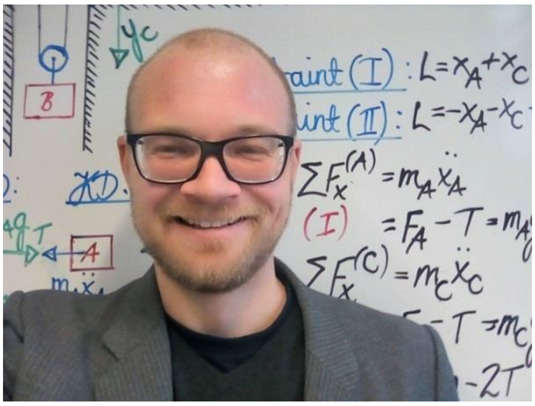
This set of lecture notes contain the main contents, which will be covered in the HSRW mechanical engineering elective on Machine Dynamics. It has, in order to try something new for once, been attempted not to include long a tedious derivations of the underlying theory, since most of this should be available already or can be found elsewhere. Instead, the current set of lecture notes are focused on problem solving, which in a machine dynamics framework mathematically speaking gets more than tedious enough.
The course is initiated with a recap of plane kinematics of rigid bodies based on the classical formulation of the principle of relative motion (see for example [1]) along with the equivalent formulation commonly applied in MBD based on transformation between a fixed global– and a moving body mounted frame, see [2] and [3]. Secondly, it will be recapped how equations of motions are derived using the Newton-Euler equations, Lagrange's principle and the conventional MBD approach to forward dynamics (usually applied for numerical analysis). Finally, before taking a brief detour into equations of motion and eigenfrequencies for multi-DOF particle vibrations, a crash course in old school rotor dynamics will be given with particular focus on whirling- and fault frequencies and their importance in machine dynamics and diagnostics and condition monitoring of rotating equipment. It has turned out to be unexpectedly difficult to recommend a good text book on these topics (this fact may suggest that we should do our own book in the future). For now, the required theoretical background is contained in [4], but students are encouraged to have a bit of fun with the fundamentals of signal processing for noise and vibrations on their own [6].

1.1. The standard formalities

If you are a lecturer …

If you are a student, remember the first three rules required for learning mechanics:

1. Come to the lectures, go to the exercises
2. Come to the lectures, go to the exercises
3. Mechanics is NOT hard, but if it does not hurt when you start learning something new, you are probably not doing it right



Regards,

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2. Plane dynamics revisited with the sliding rod problem

The first problem we will consider is the so-called sliding rod (sometimes referred to as the 'sliding ladder') problem, see Figure 1. As the matter of fact, this problem will be applied to recap and explain almost all concepts from classical plane dynamics.

2.1. Kinematics

2.1.1. Calculated example: analytical kinematics analysis rod (classical approach)

Firstly, the classical approach to rigid body kinematics based on the principle of relative motion will be recapped.

Problem description: A slender rod of length L has in each end wheels attached as shown in Figure 1. The left wheel denoted A is fixed in a vertical trace while the right end denoted B is fixed in a horizontal trace. The right end of the rod is subjected to a constant velocity v_B due to a … let's say there's an actuator acting on this end.   Determine the velocity and acceleration of wheel A along with the angular velocity and acceleration of the rod.

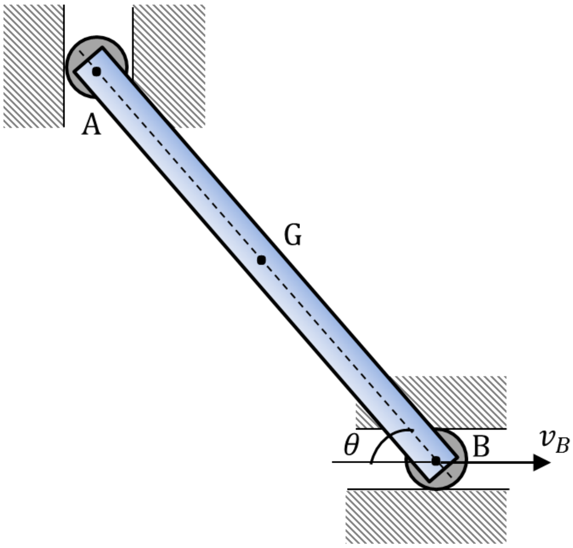


Figure 1 Sliding rod problem

Theory recap: It is recalled from classical dynamics [1], that the principle of relative motion at position level can be formulated between two points A and B as

$$\boldsymbol{r}_A = \boldsymbol{r}_B + \boldsymbol{r}_{A/B} \tag{1.}$$

In which $\boldsymbol{r}_{A/B}$ denotes the position of A relative to- or seen from B. Differentiation provides us with the following expressions:

$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{v}_{A/B} \rightarrow \boldsymbol{a}_A = \boldsymbol{a}_B + \boldsymbol{a}_{A/B} \tag{2.}$$

In which $\boldsymbol{v}_{A/B} = \boldsymbol{\omega} \times \boldsymbol{r}_{A/B}$. At acceleration level, the relative acceleration term $\boldsymbol{a}_{A/B}$ is commonly split in to a normal term $\left(\boldsymbol{a}_{A/B}\right)_n = \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \boldsymbol{r}_{A/B}\right)$ and a tangential term $\left(\boldsymbol{a}_{A/B}\right)_t = \boldsymbol{\alpha} \times \boldsymbol{r}_{A/B}$

$$\boldsymbol{a}_A = \boldsymbol{a}_B + \left(\boldsymbol{a}_{A/B}\right)_n + \left(\boldsymbol{a}_{A/B}\right)_t \tag{3.}$$

It is noted that $\boldsymbol{a}_B$ is related to a translating motion with no change of spatial orientation, while $\boldsymbol{a}_{A/B}$ describes a rotating motion. The sum of the two yields a general motion.

Solution: in order to determine the unknown velocity terms, we apply the principle of relative motion at velocity level from equation 2. It is chosen to consider the velocity of A seen from B, but this could might as well have been done the other way around. Sketching directions and magnitudes and writing down the vector equation on component form, we obtain

$\boldsymbol{v}_A$	$= \boldsymbol{v}_B$	$+ \boldsymbol{v}_{A/B}$
↓	→	↙ θ
	v_B	ωl

$$\begin{pmatrix} 0 \\ -v_A \end{pmatrix} = \begin{pmatrix} v_B \\ 0 \end{pmatrix} + \begin{pmatrix} -\omega L sin\theta \\ -\omega L cos\theta \end{pmatrix} \rightarrow \begin{cases} \omega = \dfrac{v_B}{L sin\theta} \\ v_A = \omega L cos\theta = \dfrac{v_B}{tan\theta} \end{cases} \qquad 4.$$

It is noted that $\boldsymbol{v}_{A/B}$ since it is defined as cross-product, is perpendicular to $\boldsymbol{r}_{A/B}$. Continuing in the same fashion, we apply the principle of relative motion at acceleration level from equation 3. Again, sketching directions and magnitudes, and writing down the vector equation on component form, we obtain

$\boldsymbol{a}_A$	$= \overset{0}{\cancel{\boldsymbol{a}_B}}$	$+ \left(\boldsymbol{a}_{A/B}\right)_n$	$+ \left(\boldsymbol{a}_{A/B}\right)_t$
↓	–	θ ↘	↙ θ
		$\omega^2 l$	αl

$$\begin{pmatrix} 0 \\ -a_A \end{pmatrix} = \begin{pmatrix} \omega^2 L cos\theta \\ -\omega^2 L sin\theta \end{pmatrix} + \begin{pmatrix} -\alpha L sin\theta \\ -\alpha L cos\theta \end{pmatrix} \rightarrow \begin{cases} \alpha = \omega^2 \dfrac{cos\theta}{sin\theta} = \dfrac{\omega^2}{tan\theta} \\ a_A = \omega^2 L sin\theta + \alpha L cos\theta = \omega^2 L \left(sin\theta + \dfrac{cos\theta}{tan\theta} \right) \end{cases} \qquad 5.$$

In which $\boldsymbol{a}_B = 0$, since $\boldsymbol{v}_B$ is constant. It is noted that the relative acceleration terms form the acceleration components of a point in circular motion, and that the normal term $\left(\boldsymbol{a}_{A/B}\right)_n$ points from A to B along $\boldsymbol{r}_{A/B}$ and the tangential term $\left(\boldsymbol{a}_{A/B}\right)_t$ is perpendicular to the normal direction and $\boldsymbol{r}_{A/B}$.

2.1.2. Calculated example: vector kinematics analysis (MBD approach)

Problem description: We will now solve the same problem as in the previous section (2.1.1) by using transformation matrices based on a mapping between a global fixed xy-based frame and a local $\xi\eta$-based frame.

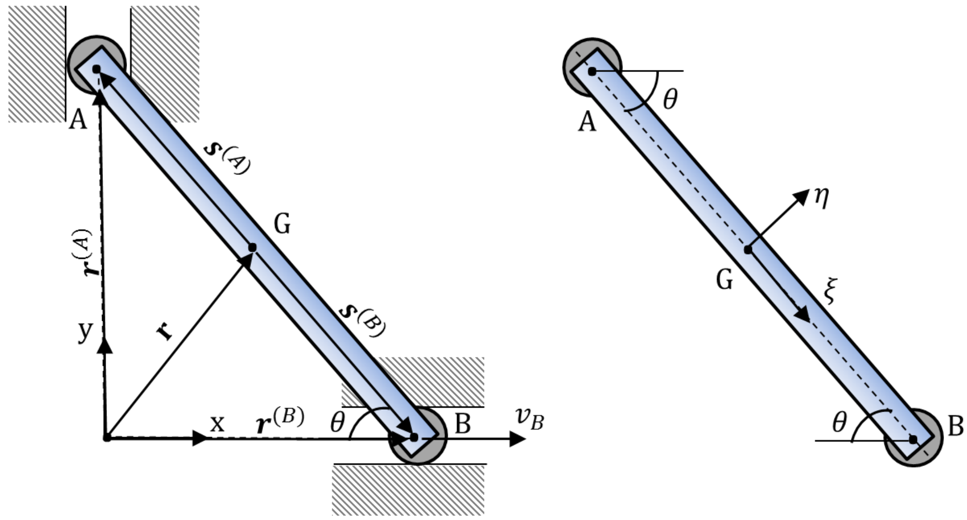


Figure 2 Vectors and coordinate systems required

Theory recap: from the theory used for computational MBD described in [2] and [3], we recall that the global position vector of any point P in a rigid body $(\boldsymbol{r})^P$ can be written as the sum of the position vector of the body C.o.G $(\boldsymbol{r})$ and the relative distance between the C.o.G and point P $(\boldsymbol{s})^P_{xy}$. The latter can be expressed in a global xy-based frames. It may however also be expressed in a local $\xi\eta$-based frame. These can be related by the transformation matrix $[\boldsymbol{A}]$, which is a function of the angle θ describing the spatial orientation of the body. **We note, that this angle is always measured with respect to the global x-axis and is positive and in counter-clockwise direction.**

$$(\boldsymbol{r})^P = (\boldsymbol{r}) + (\boldsymbol{s})^P_{xy} = (\boldsymbol{r}) + [\boldsymbol{A}](\boldsymbol{s})^P_{\xi\eta} \qquad 6.$$

The velocity and acceleration terms can be obtained by time differentiation. Recalling that $(\dot{\boldsymbol{s}})_{xy}^{P} = (\breve{\boldsymbol{s}})_{xy}^{P}\dot{\theta}$ (intuitively clear if you think about it and draw it), we obtain the velocities

$$(\dot{\boldsymbol{r}})^{P} = (\dot{\boldsymbol{r}}) + (\breve{\boldsymbol{s}})_{xy}^{P}\dot{\theta} \qquad \text{with } (\breve{\boldsymbol{s}})_{xy}^{P} = \begin{pmatrix} -s_y^P \\ s_x^P \end{pmatrix} \tag{7.}$$

Furthermore, the accelerations can be obtained as

$$(\ddot{\boldsymbol{r}})^{P} = (\ddot{\boldsymbol{r}}) + (\breve{\boldsymbol{s}})_{xy}^{P}\ddot{\theta} - (\boldsymbol{s})_{xy}^{P}\dot{\theta}^2 \tag{8.}$$

Solution: It is noted, that the spatial orientation of the rod is negative when using the common convention for transformation matrices. Initially, the coordinates of the rod end-points in the local frame are obtained as

$$(\boldsymbol{s})_{\xi\eta}^{A} = \begin{pmatrix} -\frac{L}{2} \\ 0 \end{pmatrix} \qquad (\boldsymbol{s})_{\xi\eta}^{B} = \begin{pmatrix} \frac{L}{2} \\ 0 \end{pmatrix}$$

Transformation to global coordinates can be performed applying the second term

$$(\boldsymbol{s})_{xy}^{A} = [\boldsymbol{A}](\boldsymbol{s})_{\xi\eta}^{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{pmatrix} -\frac{L}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{L}{2}\cos\theta \\ -\frac{L}{2}\sin\theta \end{pmatrix} \qquad (\boldsymbol{s})_{xy}^{B} = [\boldsymbol{A}](\boldsymbol{s})_{\xi\eta}^{B} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{pmatrix} \frac{L}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{L}{2}\cos\theta \\ \frac{L}{2}\sin\theta \end{pmatrix}$$

We may now apply equation 7 and calculate the end point velocities

$$(\dot{\boldsymbol{r}})^{A} = (\dot{\boldsymbol{r}}) + (\breve{\boldsymbol{s}})_{xy}^{A}\dot{\theta} \rightarrow \begin{pmatrix} 0 \\ v^A \end{pmatrix} = \begin{pmatrix} v_x^G \\ v_y^G \end{pmatrix} + \begin{pmatrix} \frac{L}{2}\sin\theta \\ -\frac{L}{2}\cos\theta \end{pmatrix}\dot{\theta} \qquad (\dot{\boldsymbol{r}})^{B} = (\dot{\boldsymbol{r}}) + (\breve{\boldsymbol{s}})_{xy}^{B}\dot{\theta} \rightarrow \begin{pmatrix} v^B \\ 0 \end{pmatrix} = \begin{pmatrix} v_x^G \\ v_y^G \end{pmatrix} + \begin{pmatrix} -\frac{L}{2}\sin\theta \\ \frac{L}{2}\cos\theta \end{pmatrix}\dot{\theta}$$

These two equations contain the x and y velocity components of G and the angular velocity $\dot{\theta}$ as unknowns. Isolating the velocity of G in both the left and the right equation, and setting these equal provides us with the following expression allowing for calculation of the angular velocity:

$$\begin{pmatrix} 0 \\ v^A \end{pmatrix} - \begin{pmatrix} \frac{L}{2}\sin\theta \\ -\frac{L}{2}\cos\theta \end{pmatrix}\dot{\theta} = \begin{pmatrix} v^B \\ 0 \end{pmatrix} - \begin{pmatrix} -\frac{L}{2}\sin\theta \\ \frac{L}{2}\cos\theta \end{pmatrix}\dot{\theta}$$

Considering the x-components, it follows that

$$\textbf{(I)} \rightarrow -\frac{L}{2}\sin\theta\dot{\theta} = v^B + \frac{L}{2}\sin\theta\dot{\theta} \rightarrow \dot{\theta} = -\frac{v^B}{L\sin\theta}$$

Since the orientation is negative, and $\sin(-\theta) = -\sin(\theta)$, this is the same result as obtained in the previous example with conventional principles. The linear velocity can now be obtained

$$\textbf{(II)} \rightarrow v^A = -\frac{L}{2}\cos\theta\dot{\theta} - \frac{L}{2}\cos\theta\dot{\theta} = -L\cos\theta\dot{\theta} = -L\cos\theta\left(-\frac{v^B}{L\sin\theta}\right) = \frac{v^B}{\tan\theta}$$

The acceleration can be considered in a similar fashion using equation 8.

$$(\ddot{\boldsymbol{r}})^{A} = (\ddot{\boldsymbol{r}}) + (\breve{\boldsymbol{s}})_{xy}^{A}\ddot{\theta} - (\boldsymbol{s})_{xy}^{A}\dot{\theta}^2 \rightarrow \begin{pmatrix} 0 \\ a_A \end{pmatrix} = \begin{pmatrix} a_x^G \\ a_y^G \end{pmatrix} + \begin{pmatrix} \frac{L}{2}\sin\theta \\ -\frac{L}{2}\cos\theta \end{pmatrix}\ddot{\theta} - \begin{pmatrix} -\frac{L}{2}\cos\theta \\ -\frac{L}{2}\sin\theta \end{pmatrix}\dot{\theta}^2$$

$$(\ddot{\boldsymbol{r}})^{B} = (\ddot{\boldsymbol{r}}) + (\breve{\boldsymbol{s}})_{xy}^{B}\ddot{\theta} - (\boldsymbol{s})_{xy}^{B}\dot{\theta}^2 \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_x^G \\ a_y^G \end{pmatrix} + \begin{pmatrix} -\frac{L}{2}\sin\theta \\ \frac{L}{2}\cos\theta \end{pmatrix}\ddot{\theta} - \begin{pmatrix} \frac{L}{2}\cos\theta \\ \frac{L}{2}\sin\theta \end{pmatrix}\dot{\theta}^2$$

$$\begin{aligned} \textbf{(I)} &\rightarrow a_x^G + \frac{L}{2}\sin\theta\ddot{\theta} + \frac{L}{2}\cos\theta\dot{\theta}^2 = a_x^G - \frac{L}{2}\sin\theta\ddot{\theta} - \frac{L}{2}\cos\theta\dot{\theta}^2 \\ &\rightarrow \ddot{\theta}\left(\frac{L}{2}\sin\theta + \frac{L}{2}\sin\theta\right) = -\frac{L}{2}\cos\theta\dot{\theta}^2 - \frac{L}{2}\cos\theta\dot{\theta}^2 \\ &\rightarrow \ddot{\theta} = -\frac{L\cos\theta}{L\sin\theta}\dot{\theta}^2 = -\frac{\dot{\theta}^2}{\tan\theta} \end{aligned}$$

Since the orientation is negative, and $\tan(-\theta) = -\tan(\theta)$, this is again the same result as in the previous example. Finally, the unknown acceleration can be determined

$$\begin{aligned} \textbf{(II)} &\rightarrow -a_A + a_y^G - \frac{L}{2}\cos\theta\ddot{\theta} + \frac{L}{2}\sin\theta\dot{\theta}^2 = a_y^G + \frac{L}{2}\cos\theta\ddot{\theta} - \frac{L}{2}\sin\theta\dot{\theta}^2 \\ &\rightarrow a_A = -L\cos\theta\ddot{\theta} + L\sin\theta\dot{\theta}^2 \\ &\qquad = -L\cos\theta\left(-\frac{\dot{\theta}^2}{\tan\theta}\right) + L\sin\theta\dot{\theta}^2 \\ &\qquad = \dot{\theta}^2 L\left(\sin\theta + \frac{\cos\theta}{\tan\theta}\right) \end{aligned}$$

This result also corresponds to what was obtained in the previous example.
Clearly, the latter approach has not proven more efficient for analytical calculations. However, it turns out to be more convenient for implementation in numerical models than the classical approach, and is in additional quite practical when formulating kinematic constrains parametrically for general purpose codes, see [2] and [3].

2.2. Calculated example: forward dynamics

A long and slender rod with geometry as in the previous examples is considered. However, the rod is now in the lower end B subjected to an end force P and as consequence, the motion of the rod can no longer be determined only based on kinematics, see Figure 3. I.e. we are going to need the Newton-Euler equations.
In this section, a rod with length $L = 0.8$ m, mass $m = 12$ kg, initial angle $\theta_0 = 60$ deg and external end load P=100 N.

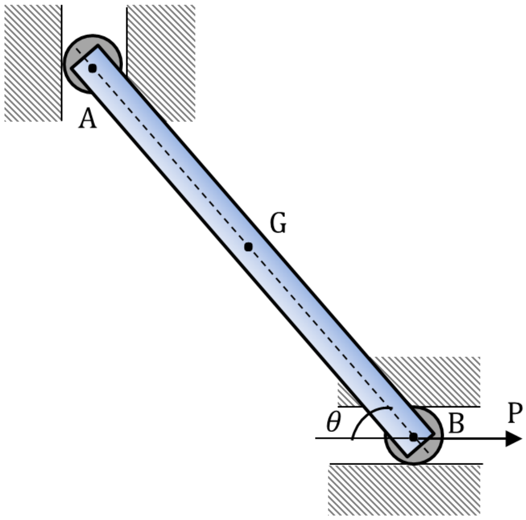
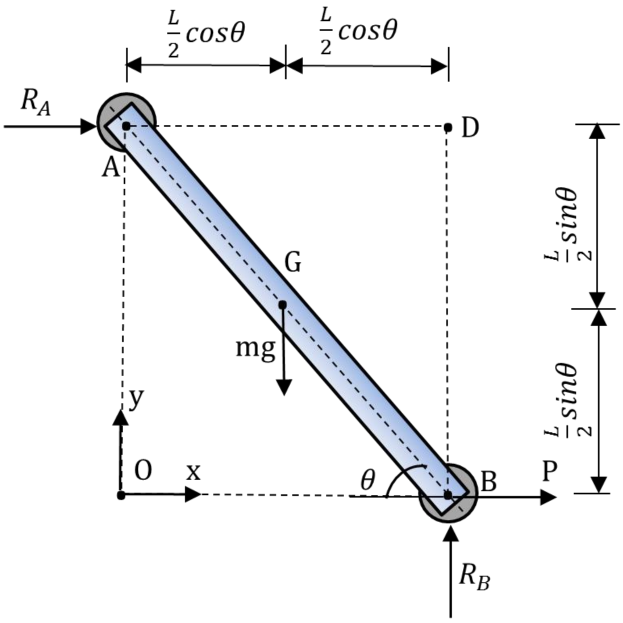


Figure 3 Sliding rod with end load

2.2.1. Newton-Euler approach to solution for forces for initial conditions

Problem description: Initially, we will consider the rod as it is being released from rest, i.e. possesses no velocity components. In this scenario, the objective is now to determine the angular acceleration α and the reaction forces R_A and R_B.



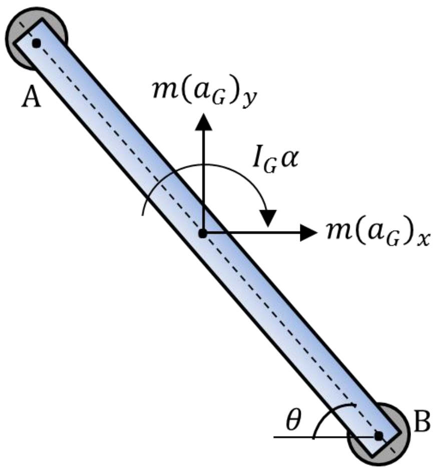


Figure 4 Free-body diagram (FBD)

Figure 5 Kinetic diagram (KD)

Theory recap: From basic dynamics, it is recalled that the dynamics of rigid bodies are governing by the Newton-Euler equations

$$\sum F_x = m\ddot{x} \qquad \sum F_y = m\ddot{y} \qquad \sum M = I\ddot{\theta} \tag{9.}$$

A free-body diagram (FB) and a kinetic diagram (KD) would usually be applied to sketch the left and right hand side of the equations in order to ensure that force and kinematics properties are considered with the same sign conventions.

Solution: Considering the FBD in Figure 4 and the KD in Figure 5, the force equilibriums arising from Newton's 2nd law become

$$\sum F_x = ma_x = R_A + P \tag{10.}$$

$$\sum F_y = ma_y = R_B - mg \tag{11.}$$

The moment equilibrium arising from Euler's law of motion can be formulated around the C.o.G:

$$\sum M_G = I_G\alpha = R_A\frac{L}{2}sin\theta - R_B\frac{L}{2}cos\theta - P\frac{L}{2}sin\theta \tag{12.}$$

Equations 10,11 and 12 yield three equations with five unknowns, a_x, a_y, α, R_A and R_B requiring us to come up with two algebraic equations based on kinematic constraints. However, if the parallel axis theorem is applied, the moment equilibrium can be formulated around any point. We will choose 'the magic point' D at the intersection point between the lines of actions of the reaction forces, eliminating those from the equations

$$\sum M_D = I_G\alpha - m(a_G)_x\frac{L}{2}sin\theta + m(a_G)_y\frac{L}{2}cos\theta = -mg\frac{L}{2}cos\theta - PLsin\theta \tag{13.}$$

This is the moment equilibrium which algebraically speaking is easiest to handle. In order to formulate kinematic constraints linking the kinematic parameters, the principle of relative motion from equation 3. Is applied at acceleration level

$\boldsymbol{a}_A$	$=\ \boldsymbol{a}_B$	$+\ (\boldsymbol{a}_{A/B})_n$	$+\ (\boldsymbol{a}_{A/B})_t$
$\downarrow$	$\rightarrow$	$\theta \searrow$	$\theta \nearrow$
		0	αl

$$\begin{pmatrix} 0 \\ -a_A \end{pmatrix} = \begin{pmatrix} a_B \\ 0 \end{pmatrix} + \begin{pmatrix} \alpha L sin\theta \\ \alpha L cos\theta \end{pmatrix} \rightarrow \begin{cases} a_A = -\alpha L cos\theta \\ a_B = -\alpha L sin\theta \end{cases} \tag{14.}$$

That didn't entirely do the job, since we're actually after the acceleration of the C.o.G. So we need to have another go

$\boldsymbol{a}_G$	$=\ \boldsymbol{a}_B$	$+\ (\boldsymbol{a}_{G/B})_n$	$+\ (\boldsymbol{a}_{G/B})_t$
	$\rightarrow$	$\theta \searrow$	$\theta \nearrow$
		0	$\alpha\frac{l}{2}$

$$\begin{pmatrix} (a_G)_x \\ (a_G)_y \end{pmatrix} = \begin{pmatrix} a_B \\ 0 \end{pmatrix} + \begin{pmatrix} \omega^2\frac{L}{2}cos\theta \\ -\omega^2\frac{L}{2}sin\theta \end{pmatrix} + \begin{pmatrix} \alpha\frac{L}{2}sin\theta \\ \alpha\frac{L}{2}cos\theta \end{pmatrix}$$

$$\rightarrow \begin{cases} (a_G)_x = a_B + \alpha\dfrac{L}{2}sin\theta = -\alpha\dfrac{L}{2}sin\theta \\ (a_G)_y = \alpha\dfrac{L}{2}cos\theta \end{cases} \tag{15.}$$

Now, algebraic equations linking a_x, a_y and α have been established. Substituting those into the moment equilibrium equation in equation 13. an expression for the angular acceleration α is obtained

$$\begin{aligned}
\sum M_D &= I_G\alpha - m(a_G)_x\frac{L}{2}sin\theta + m(a_G)_y\frac{L}{2}cos\theta = -mg\frac{L}{2}cos\theta - PLsin\theta \\
&= \left(\frac{1}{12}mL^2\right)\alpha - m\left(-\alpha\frac{L}{2}sin\theta\right)\frac{L}{2}sin\theta + m\left(\alpha\frac{L}{2}cos\theta\right)\frac{L}{2}cos\theta \\
&= \left(\frac{1}{12}mL^2\right)\alpha + m\frac{L^2}{4}(\alpha sin^2\theta + \alpha cos^2\theta) \\
&= \left(\frac{1}{12}mL^2\right)\alpha + m\frac{L^2}{4}\alpha \\
&= \left(\frac{1}{12} + \frac{1}{4}\right)mL^2\alpha = \frac{mL^2}{3}\alpha \\
&\rightarrow \frac{mL^2}{3}\alpha + mg\frac{L}{2}cos\theta + PLsin\theta = 0 \\
&\rightarrow \alpha = -\frac{3}{mL^2}\left(mg\frac{L}{2}cos\theta + PLsin\theta\right) = -36.3\frac{rad}{s^2}
\end{aligned} \tag{16.}$$

This gives us the following acceleration components

$(a_G)_x = -\alpha \frac{L}{2} sin\theta = 12.6 \frac{m}{s^2}$ 17.

$(a_G)_y = \alpha \frac{L}{2} cos\theta = \ -7.3 \frac{m}{s^2}$ 18.

Now returning to the force equilibrium equations, we obtain the following two expressions for the reaction forces

$R_A = ma_x - P = \quad 50.7\ N$ 19.

$R_B = ma_y + \ mg = 30.7\ N$ 20.

This has provided us with the angular acceleration and the reaction forces at the moment the rod is released

2.2.2. Newton-Euler approach to EoM

Problem description: The sliding rod with an end force considered in 2.2.1 will again be subject of analysis with the general equation of motion as objective.

Solution: since we already figured out that the kinematics are required during our first take on this problem, we might as well start out straightening this out. In order to obtain kinematic constraints, the principle of relative motion will again be applied at acceleration level. However, the key difference from the previous problem is, that the angular acceleration no longer can be set to zero, since we're not only after the initial values as the rod is released, but after the general equations of motion.

$\boldsymbol{a}_A =$	$\boldsymbol{a}_B +$	$(\boldsymbol{a}_{A/B})_n +$	$(\boldsymbol{a}_{A/B})_t$
$\downarrow$	$\rightarrow$	$\theta \searrow$	$\theta \nearrow$
		$\omega^2 l$	αl

$$\begin{pmatrix} 0 \\ -a_A \end{pmatrix} = \begin{pmatrix} a_B \\ 0 \end{pmatrix} + \begin{pmatrix} \omega^2 Lcos\theta \\ -\omega^2 Lsin\theta \end{pmatrix} + \begin{pmatrix} \alpha Lsin\theta \\ \alpha Lcos\theta \end{pmatrix} \qquad 21.$$

$$\rightarrow \begin{cases} a_A = \omega^2 Lsin\theta - \alpha Lcos\theta \\ a_B = -\omega^2 Lcos\theta - \alpha Lsin\theta \end{cases}$$

Again, going for the acceleration of the center of gravity, we obtain

$\boldsymbol{a}_G =$	$\boldsymbol{a}_B +$	$(\boldsymbol{a}_{G/B})_n +$	$(\boldsymbol{a}_{G/B})_t$
	$\rightarrow$	$\theta \searrow$	$\theta \nearrow$
		$\omega^2 \frac{l}{2}$	$\alpha \frac{l}{2}$

$$\begin{pmatrix} (a_G)_x \\ (a_G)_y \end{pmatrix} = \begin{pmatrix} a_B \\ 0 \end{pmatrix} + \begin{pmatrix} \omega^2 \frac{L}{2} cos\theta \\ -\omega^2 \frac{L}{2} sin\theta \end{pmatrix} + \begin{pmatrix} \alpha \frac{L}{2} sin\theta \\ \alpha \frac{L}{2} cos\theta \end{pmatrix} \qquad 22.$$

$$\rightarrow \begin{cases} (a_G)_x = a_B + \omega^2 \frac{L}{2} cos\theta + \alpha \frac{L}{2} sin\theta \\ (a_G)_y = -\omega^2 \frac{L}{2} sin\theta + \alpha \frac{L}{2} cos\theta \end{cases}$$

Now, the following two expressions for the linear accelerations of the C.o.G. can be obtained in terms of the angular acceleration α

$$\begin{aligned} (a_G)_x &= a_B + \omega^2 \frac{L}{2} cos\theta + \alpha \frac{L}{2} sin\theta \qquad 23. \\ &= (-\omega^2 Lcos\theta - \alpha Lsin\theta) + \omega^2 \frac{L}{2} cos\theta + \alpha \frac{L}{2} sin\theta \\ &= -\omega^2 \frac{L}{2} cos\theta - \alpha \frac{L}{2} sin\theta \end{aligned}$$

$$(a_G)_y = -\omega^2 \frac{L}{2} sin\theta + \alpha \frac{L}{2} cos\theta \qquad 24.$$

the FBD in Figure 4 and the KD in Figure 5 are still valid. Applying the moment equilibrium, we obtain

$$\begin{aligned} \sum M_D &= I_G \alpha - m(a_G)_x \frac{L}{2} sin\theta + m(a_G)_y \frac{L}{2} cos\theta = -mg\frac{L}{2} cos\theta - PLsin\theta \\ &= \left(\frac{1}{12} mL^2\right) \alpha - m\left(-\omega^2 \frac{L}{2} cos\theta - \alpha \frac{L}{2} sin\theta\right) \frac{L}{2} sin\theta + m\left(-\omega^2 \frac{L}{2} sin\theta + \alpha \frac{L}{2} cos\theta\right) \frac{L}{2} cos\theta \\ &= \left(\frac{1}{12} mL^2\right) \alpha + m\frac{L^2}{4}(\omega^2 cos\theta sin\theta + \alpha sin^2\theta - \omega^2 sin\theta cos\theta + \alpha cos^2\theta) \\ &= \left(\frac{1}{12} mL^2\right) \alpha + m\frac{L^2}{4} \alpha \qquad 25. \\ &= \left(\frac{1}{12} + \frac{1}{4}\right) mL^2 \alpha = \frac{mL^2}{3} \alpha \\ &\rightarrow \frac{mL^2}{3} \alpha + mg\frac{L}{2} cos\theta + PLsin\theta = 0 \\ &\rightarrow \alpha + \frac{3g}{2L} cos\theta + \frac{3P}{mL} sin\theta = 0 \qquad \alpha = \ddot{\theta} \end{aligned}$$

$$\rightarrow \ddot{\theta} + \frac{3g}{2L}cos\theta + \frac{3P}{mL}sin\theta = 0$$

This is the equation of motion. It is more than just a tiny bit non-linear and we do not have a standard solution form available. However, it will later in section 2.2.5 show how these can be integrated numerically.

2.2.3. Lagrandian approach to EoM

Problem description: Again, the sliding rod problem from section 2.2.2 is considered with the equation of motion as objective. However, this time Lagrandian dynamics will be applied to obtain the equations of motion.

Theory recap: When applying Lagrandian mechanics to obtain EoM, a set of generalized coordinates y_i are chosen. On basis of the Lagrandian given by the kinetic energy minus the potential energy, the dynamic equilibrium can be determined as the stationary state of the Lagrandian. Using calculus of variations, it can be shown, that this state is found as the state fulfilling the Euler-Lagrange equation

$$\frac{\partial L}{\partial y_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right) = 0 \qquad \text{with Lagrandian } L = T - U \tag{26.}$$

It is recalled that the kinetic energy for a rigid body is given by

$$T_{translation} = \frac{1}{2}mv^2 \qquad T_{rotation} = \frac{1}{2}I_G\omega^2 \qquad T = T_{rotation} + T_{translation} \tag{27.}$$

While the work of external forces and moments are given by

$$W_P = \int Pdx \qquad W_M = \int Md\theta \tag{28.}$$

This work is related to potential energies for conservative force fields by

$$U_P = -W_P \qquad U_M = -W_M \qquad U = U_P + U_M \tag{29.}$$

Solution: we will chose θ as generalized coordinate. This enables us to rewrite the coordinates of the C.o.G. as

$$x_G = \frac{L}{2}cos\theta \qquad y_G = \frac{L}{2}sin\theta \tag{30.}$$

Time differentiation provides the component wise velocities

$$\dot{x}_G = -\frac{L}{2}sin\theta \cdot \dot{\theta} \qquad \dot{y}_G = \frac{L}{2}cos\theta \cdot \dot{\theta} \tag{31.}$$

The kinetic energy is now given by

$$\begin{aligned}
T &= \frac{1}{2}mv^2 + \frac{1}{2}I_G\omega^2 \\
&= \frac{1}{2}m((\dot{x}_G)^2 + (\dot{y}_G)^2) + \frac{1}{2}I_G\dot{\theta}^{\,2} \\
&= \frac{1}{2}m\left(\left(-\frac{L}{2}sin\theta \cdot \dot{\theta}\right)^2 + \left(\frac{L}{2}cos\theta \cdot \dot{\theta}\right)^2\right) + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^{\,2} \\
&= \frac{1}{2}m\left(\frac{L}{2}\right)^2 \dot{\theta}^2(sin^2\theta + cos^2\theta) + \frac{1}{24}(mL^2)\dot{\theta}^{\,2} \\
&= \frac{1}{2}m\left(\frac{L}{2}\right)^2 \dot{\theta}^2(sin^2\theta + cos^2\theta) + \frac{1}{24}(mL^2)\dot{\theta}^{\,2} \\
&= m\frac{L^2}{8}\dot{\theta}^2 + m\frac{L^2}{24}\dot{\theta}^2 \\
&= m\frac{L^2}{6}\dot{\theta}^2
\end{aligned} \tag{32.}$$

The potential energy is

$$\begin{aligned}
U &= mgy_G - \int Pdx - \int Md\theta \\
&= mg\left(\frac{L}{2}sin\theta\right) - Px_G - \int\left(-P\frac{L}{2}sin\theta\right)d\theta \\
&= mg\left(\frac{L}{2}sin\theta\right) - P\left(\frac{L}{2}cos\theta\right) - \left(P\frac{L}{2}cos\theta\right) \\
&= mg\left(\frac{L}{2}sin\theta\right) - PLcos\theta
\end{aligned} \tag{33.}$$

We may now define the Lagrandian

$$L = T - U$$
$$= m\frac{L^2}{6}\dot{\theta}^2 - \left(mg\left(\frac{L}{2}sin\theta\right) - PLcos\theta\right) \qquad 34.$$
$$= m\frac{L^2}{6}\dot{\theta}^2 - mg\left(\frac{L}{2}sin\theta\right) - PLcos\theta$$

Applying the Euler-Lagrange theorem for one variable, the following is obtained

$$\frac{\partial L}{\partial \theta} = -mg\frac{L}{2}cos\theta - PLsin\theta = -mg\frac{L}{2}cos\theta - PLsin\theta$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(\frac{2}{6}mL^2\dot{\theta}\right) = \frac{1}{3}mL^2\ddot{\theta} \qquad 35.$$
$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = -mg\frac{L}{2}cos\theta - PLsin\theta - \frac{1}{3}mL^2\ddot{\theta} = 0$$
$$\rightarrow \ddot{\theta} + \frac{3}{2}\frac{g}{L}cos\theta + \frac{3P}{mL}sin\theta = 0 \qquad 36.$$

This obviously corresponds to the EoM obtained in section 2.2.2.

2.2.4. Computational MBD approach

Problem description: We will now, for the last time consider the sliding rod with a constant end load from section 2.2.2, but this time we will base our analysis of the problem on methods for forward dynamics analysis of MBD systems from [2] and [3].

Theory recap: We recall that forward dynamics analysis of constrained systems is performed based on a set of kinematic constraints on the form

$$(\boldsymbol{\Phi}) = 0 \qquad 37.$$

These can be considered related to the choice of generalized coordinates in Lagrandian mechanics (for example equation 30), but apply for a point where constrains are added. On this basis we define the following quantities

$$(\dot{\boldsymbol{\Phi}}) = [\boldsymbol{D}](\dot{\boldsymbol{q}}) = \mathbf{0} \qquad (\boldsymbol{\gamma}) = -[\dot{\boldsymbol{D}}](\dot{\boldsymbol{q}}) \qquad 38.$$

in which $[\boldsymbol{D}]$ is the Jacobian matrix and $(\boldsymbol{q})$ is a set of generalized coordinates related to the body C.o.G. We will iteratively, by looping through time solve, the following linear system for each time step

$$\begin{bmatrix} [\boldsymbol{M}] & [\boldsymbol{D}]^{\boldsymbol{T}} \\ [\boldsymbol{D}] & \mathbf{0} \end{bmatrix}\begin{pmatrix} (\ddot{\boldsymbol{q}}) \\ (\boldsymbol{\lambda}) \end{pmatrix} = \begin{pmatrix} (\boldsymbol{h}) \\ (\boldsymbol{\gamma}) \end{pmatrix} \qquad 39.$$

In which $(\boldsymbol{h})$ contains external forces and $[\boldsymbol{M}]$ ist he body mass matrix including the body inertia. This will provide the generalized accelerations $(\ddot{\boldsymbol{q}})$ and the corresponding velocities and accelerations can be found by numerical integration, see section 2.2.5.

Solution: In order to fix the right and left end of the rod in the traces, the following two kinematic constrains are formulated and arranged onto the form applied in equation 37

$$\left.\begin{matrix} x_A = 0: x_G = \frac{L}{2}cos\theta \\ y_B = 0: y_G = \frac{L}{2}sin\theta \end{matrix}\right\} \rightarrow \mathbf{\Phi} = \begin{pmatrix} x_G - \frac{L}{2}cos\theta \\ y_G - \frac{L}{2}sin\theta \end{pmatrix} = 0 \qquad 40.$$

The Jacobian matrix can now in accordance applying equation 38

$$\dot{\mathbf{\Phi}} = \begin{pmatrix} \dot{x}_G + \frac{L}{2}sin\theta \cdot \dot{\theta} \\ \dot{y}_G - \frac{L}{2}cos\theta \cdot \dot{\theta} \end{pmatrix} = 0 \rightarrow \begin{bmatrix} 1 & 0 & \frac{L}{2}sin\theta \\ 0 & 1 & -\frac{L}{2}cos\theta \end{bmatrix}\begin{pmatrix} \dot{x}_G \\ \dot{y}_G \\ \dot{\theta} \end{pmatrix} = 0 \qquad 41.$$

We now have

$$[\boldsymbol{D}] = \begin{bmatrix} 1 & 0 & \frac{L}{2}sin\theta \\ 0 & 1 & -\frac{L}{2}cos\theta \end{bmatrix} \rightarrow [\dot{\boldsymbol{D}}] = \begin{bmatrix} 0 & 0 & \frac{L}{2}cos\theta \\ 0 & 0 & \frac{L}{2}sin\theta \end{bmatrix} \qquad 42.$$

The remaining vectors and matrices required to solve equation 39 are now given by

$$[\boldsymbol{M}] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_G \end{bmatrix} \qquad (\boldsymbol{h}) = \begin{pmatrix} P \\ -mg \\ -P\frac{L}{2}sin\theta \end{pmatrix} \qquad (\boldsymbol{q}) = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \quad 43.$$

The numerical implementation of the derived equations is demonstrated in section 2.2.5.

2.2.5. Comparison of results: forward dynamics

Time integration of the analytical equations of motion 25 and 36 can be performed with various techniques. The most common solver for numerical solution of differential equations in time is the 4th order Runge-Kutta (in Matlab/Octave called ode45) with variable time steps. However, it is in many contexts of importance to write a simply solver. The first order Implicit-Euler scheme is often useful though requiring very small time steps, due to it's simplicity.

$$v_{i+1} = v_i + a_i \Delta t \quad 44.$$

$$r_{i+1} = r_i + v_{i+1} \Delta t \quad 45.$$

It is noted, that the updated velocity is used for updating the position making the method implicit. This will for most applications improve the accuracy from terribly (as in the Explicit-Euler algorithm) to poor, but sufficient when using small time steps.
The implementation of the MBD method from the previous example is shown in the following Matlab code example. Setting the length L=0.8 m, the mass m=12 kg and the initial angle $\theta_i = 60\,deg$, a solution can be obtained by numerical integration, see Figure 6 and Figure 7. As expected, the obtained solutions for the angle of the rod in time can be observed to correspond.

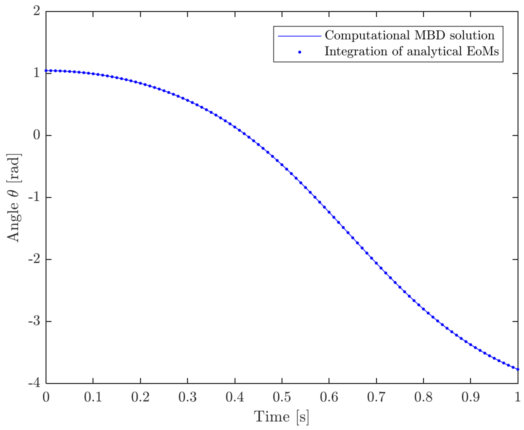


Figure 6, Solution for $P = 0$

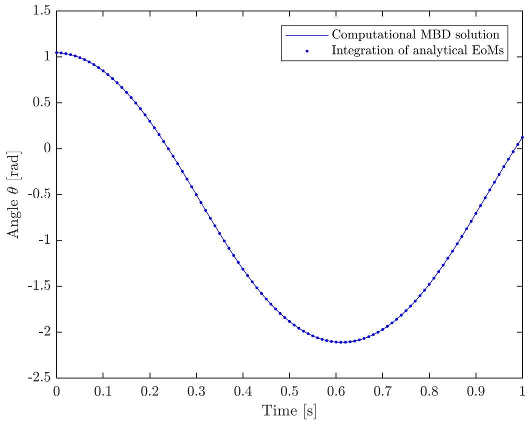


Figure 7, Solution for $P = 100$

```
%Sliding rod with end load, NEO, HSRW, 02.07.18

clc; close all; clear all;
%Model input section
L=0.8;          %Rod length
m=12;           %Rod mass
theta_ini=60;   %Initial angle
P=100;          %End load
g=9.81;         %Gravity
%Time integration section
t_lim=1;        %Simulation time limit
n=1000;         %Number of time integration points
%Setup mass matrix
M(3,3)=0; M(1,1)=m; M(2,2)=m; M(3,3)=1/12*m*L^2;
%Setup time vector
for i=1:n+1; t(i)=t_lim/n*(i-1); end; dt=t(2)-t(1);
%Initialize position and velocity arrays
q(1:3,n+1)=0; v(1:3,n+1)=0;
%Set initial position values
q(1:3,1)=[L/2*cosd(theta_ini);L/2*sind(theta_ini); theta_ini*pi/180];
%Initialize Jacobian and it's time derivative
D(2,3)=0;   D(1,1)=1;   D(2,2)=1;   dD(2,3)=0;

for i=2:n+1;        %Time integration section
    D(1,3)=L/2*sin(q(3,i-1));              D(2,3)=-L/2*cos(q(3,i-1));
    dD(1,3)=L/2*cos(q(3,i-1))*v(3,i-1);    dD(2,3)=L/2*sin(q(3,i-1))*v(3,i-1);
    F=[P;-g*m;-P*L/2*sin(q(3,i-1))];       %Load vector
    gamma=-dD*v(1:3,i-1);                  %Gamma array
    X=inv([M,D';D,zeros(2,2)])*[F;gamma];  %Solve for accelerations
    v(1:3,i)=v(1:3,i-1)+X(1:3)*dt;         %Integrate acc.
    q(1:3,i)=q(1:3,i-1)+v(1:3,i)*dt;       %Integrate vel.
    R1(i)=X(4); R2(i)=X(5);                %Reaction forces
end

%Integration of analytica EoM
theta_a(n+1)=0; omega_a(n+1)=0; theta_a(1)=theta_ini*pi/180;
for i=2:n+1;
    alpha_a=-3*g/(2*L)*cos(theta_a(i-1))-3*P/(m*L)*sin(theta_a(i-1))
    omega_a(i)=omega_a(i-1)+alpha_a*dt;
    theta_a(i)=theta_a(i-1)+omega_a(i)*dt;
end

figure; plot(t,q(3,:),'b',t,theta_a,'b.'); %Plot solutions
legend('Computational MBD solution','Integration of analytical EoMs')
xlabel('Time [s]'); ylabel('Angle $\theta$ [rad]')
```

2.3. Calculated example: inverse dynamics with computational MBD

Problem description: It turns out, that we are not entirely done with the sliding rod, since we have not yet considered how to conduct inverse dynamics analysis using computational MBD. It is recalled, that inverse dynamics refers to analyses, where a pre-defined motion is used as basis for calculated of the forces required to obtain this particular motion. We will consider the same case as analyzed in section 2.1.1, where the lower B end is subjected to a constant linear velocity.

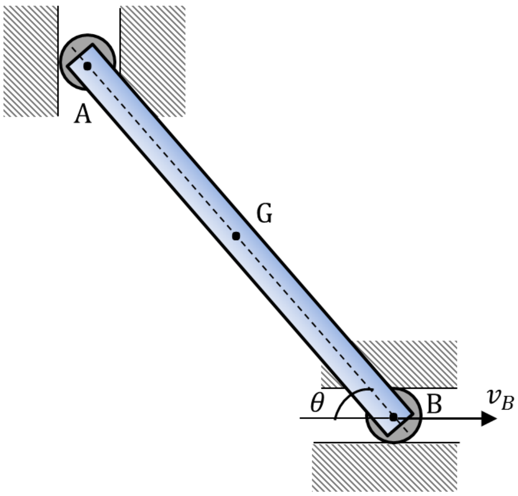


Figure 8 Inverse dynamics of a sliding rod with constant lower end velocity

Theory recap: The theoretical background for inverse dynamics is described in detail by Nikravesh in [2] under the name *the appended constraint method*. It is based on the formulation of driver constraints additional to the kinematic constraints derived in section 2.2.4. The driver constraints defines the required motion and are appended to the constraint array and therefore handled in largely the same fashion as the kinematic constraints as shown below

$$ {}^{(d)}\boldsymbol{\Phi}(\boldsymbol{q}) - \boldsymbol{f}(t) = 0 \rightarrow \begin{pmatrix} \boldsymbol{\Phi}(\boldsymbol{q}) \\ {}^{(d)}\boldsymbol{\Phi}(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{f}(t) \end{pmatrix} \qquad 46. $$

Differentiation of this equation, gives us

$$ {}^{(d)}\dot{\boldsymbol{\Phi}}(\boldsymbol{q}) = {}^{(d)}[\boldsymbol{D}](\dot{\boldsymbol{q}}) - \dot{\boldsymbol{f}}(t) = 0 \rightarrow \begin{bmatrix} [\boldsymbol{D}] \\ {}^{(d)}[\boldsymbol{D}] \end{bmatrix} (\dot{\boldsymbol{q}}) = \begin{pmatrix} \boldsymbol{0} \\ \dot{\boldsymbol{f}}(t) \end{pmatrix} \qquad 47. $$

$$ {}^{(d)}[\boldsymbol{D}](\ddot{\boldsymbol{q}}) + {}^{(d)}[\dot{\boldsymbol{D}}](\dot{\boldsymbol{q}}) - \dot{\boldsymbol{f}}(t) = 0 \rightarrow \begin{bmatrix} [\boldsymbol{D}] \\ {}^{(d)}[\boldsymbol{D}] \end{bmatrix} (\ddot{\boldsymbol{q}}) = \begin{pmatrix} -[\dot{\boldsymbol{D}}](\dot{\boldsymbol{q}}) \\ -{}^{(d)}[\dot{\boldsymbol{D}}](\dot{\boldsymbol{q}}) + \dot{\boldsymbol{f}}(t) \end{pmatrix}^{1} \qquad 48. $$

These equations can be observed to be on the conventional form we have applied for computational MBD this far. The constraint equations at position level in equation 46 are in general non-linear and requires analytical solutions or application of a numerical solver for non-linear algebraic equations like for example the Newton-Raphson method. That's the bad news. The good news are that the constraint equations at velocity and acceleration level in equation 47 and 48 are linear after a solution at position level has been obtained and can be solved using conventional linear algebra. The forces required to obtain the motion can be obtained by solving

$$ \left[[\boldsymbol{D}]^T \; {}^{(d)}[\boldsymbol{D}]^T \right] \begin{pmatrix} (\boldsymbol{\lambda}) \\ {}^{(d)}(\boldsymbol{\lambda}) \end{pmatrix} = [\boldsymbol{M}](\ddot{\boldsymbol{q}}) - (\boldsymbol{h}) \qquad 49. $$

in which $(\boldsymbol{\lambda})$ are the reactions forces and ${}^{(d)}(\boldsymbol{\lambda})$ contains the driver forces.

Solution: Applying the framework described above the sliding rod problem, the driver constraint can be formulated as

$$ x_B - f(t) = 0 \colon x_G + \frac{L}{2} cos\theta - f(t) \quad \text{with } f(t) = v_B t + L\cos\theta \qquad 50. $$

[1] For reasons yet very unclear to your mechanics professor, Nikravesh himself claims that the driver Jacobian time derivative for most practical problems is zero and presents this equation on the form

$$ \begin{bmatrix} [\boldsymbol{D}] \\ {}^{(d)}[\boldsymbol{D}] \end{bmatrix} (\ddot{\boldsymbol{q}}) = \begin{pmatrix} -[\dot{\boldsymbol{D}}](\dot{\boldsymbol{q}}) \\ \dot{\boldsymbol{f}}(t) \end{pmatrix} $$

It has not yet been possible to identify a problem where this actually is the case and the full form in equation 48 should be applied.

All constraints for the motion of the rod can now using equation 46 be formulated as

$$\begin{pmatrix} \boldsymbol{\Phi}(\boldsymbol{q}) \\ {}^{(d)}\boldsymbol{\Phi}(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} x_G - \frac{L}{2}\cos\theta \\ y_G - \frac{L}{2}\sin\theta \end{pmatrix} \\ x_G + \frac{L}{2}\cos\theta - v_B t - L\cos\theta \end{pmatrix} = \mathbf{0} \qquad 51.$$

These equations are non-linear in x_G, y_G and θ. For the sake of simplicity, we will solve those analytically by which we obtain

$$\theta = \mathrm{acos}(cos\theta_0 + v_B t) \qquad x_G = \frac{L}{2}\cos\theta \qquad y_G = \frac{L}{2}\sin\theta$$

The total Jacobian can be obtained by differentiation with respect to time

$$\begin{pmatrix} \dot{\boldsymbol{\Phi}}(\boldsymbol{q}) \\ {}^{(d)}\dot{\boldsymbol{\Phi}}(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \dot{x}_G + \frac{L}{2}\sin\theta\dot{\theta} \\ \dot{y}_G - \frac{L}{2}\cos\theta\dot{\theta} \end{pmatrix} \\ \dot{x}_G - \frac{L}{2}\sin\theta\dot{\theta} - v_B \end{pmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{L}{2}\sin\theta \\ 0 & 1 & -\frac{L}{2}\cos\theta \\ 1 & 0 & -\frac{L}{2}\sin\theta \end{bmatrix} \begin{pmatrix} \dot{x}_G \\ \dot{y}_G \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ v_B \end{pmatrix} \qquad 52.$$

We now have the following matrices, allowing us to solve equation 47 and 48 using linear algebra

$$\begin{bmatrix} [\boldsymbol{D}] \\ {}^{(d)}[\boldsymbol{D}] \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{L}{2}\sin\theta \\ 0 & 1 & -\frac{L}{2}\cos\theta \\ 1 & 0 & -\frac{L}{2}\sin\theta \end{bmatrix} \rightarrow \begin{bmatrix} [\dot{\boldsymbol{D}}] \\ {}^{(d)}[\dot{\boldsymbol{D}}] \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{L}{2}\cos\theta\dot{\theta} \\ 0 & 0 & \frac{L}{2}\sin\theta\dot{\theta} \\ 0 & 0 & -\frac{L}{2}\cos\theta\dot{\theta} \end{bmatrix} \qquad 53.$$

This allows us to solve equation 49 and obtain the reactions and driver force applying gravity as only external force on basis of the following input

$$[\boldsymbol{M}] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_G \end{bmatrix} \qquad (\boldsymbol{h}) = \begin{pmatrix} 0 \\ -mg \\ 0 \end{pmatrix} \qquad (\boldsymbol{q}) = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \qquad 54.$$

2.3.1. Comparison of results: inverse dynamics

The obtained results can be compared to the analytical model on basis of the angular velocity and acceleration derived in the previous sections

$$\omega = -\frac{v_B}{Lsin\theta} \qquad \alpha = -\omega^2\frac{cos\theta}{sin\theta} = -\frac{\omega^2}{tan\theta}$$

In which the negative signs account for that positive now is defined as the clockwise direction for increasing θ. The linear acceleration components of the center of gravity were determined to be

$$(a_G)_x = a_B + \omega^2\frac{L}{2}cos\theta + \alpha\frac{L}{2}sin\theta \qquad (a_G)_y = -\omega^2\frac{L}{2}sin\theta + \alpha\frac{L}{2}cos\theta$$

The driving force can be obtained directly from the moment equilibrium

$$\sum M_D = I_G\alpha - m(a_G)_x\frac{L}{2}sin\theta + m(a_G)_y\frac{L}{2}cos\theta = -mg\frac{L}{2}cos\theta - PLsin\theta$$

And the reaction forces are finally calculated from Newton's 2nd law

$$\sum F_x = ma_x = R_A + P \qquad \sum F_y = ma_y = R_B - mg$$

The results obtained can be compared with the calculated (λ)'s, see Figure 9 and Figure 10 on basis of the code implementation shown on the following page. The results can as expected be observed to correspond. In principle, the calculated driver force in Figure 9 could be applied as input for a forward dynamics analysis using the method introduced in section 2.2.4 and would produce results where B moves at constant velocity.

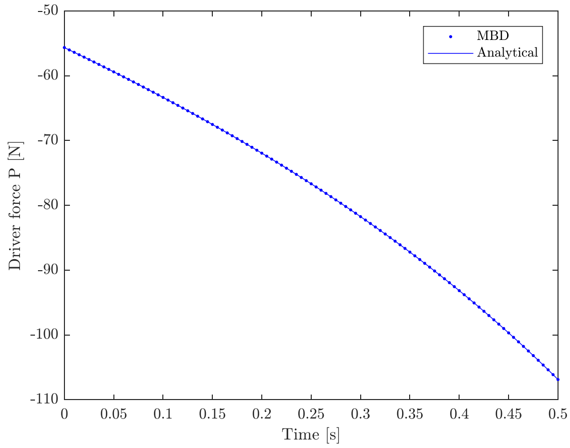


Figure 9, Example of results, driver force P

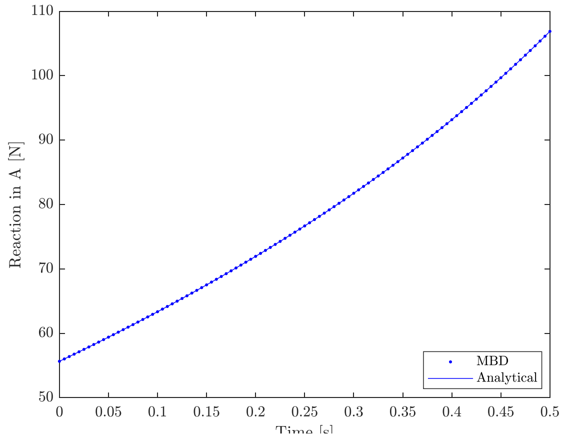


Figure 10, Example of results, reaction R_A

```
%NHS/Liding Zhang-HSRW
%Inverse dynamics of the sliding rod problem, 08.10.2020
clc; close all; clear all;
l=1.5;               %Rod length
m=20;                %Rod mass
g=9.81;              %Gravity
theta0=60*pi/180;    %Initial angle
v0=0.5;              %Velocity, B
n=100;               %Number of time steps
tlim=0.5;            %Time limit
%Setup time vectors
for i=1:n+1;t(i)=tlim/n*(i-1);end; dt=t(2)-t(1);
%Initialize arrays
q(n+1,3)=0; dq(n+1,3)=0; ddq(n+1,3)=0;
qa(n+1,3)=0; dqa(n+1,3)=0; ddqa(n+1,3)=0;
%Define mass matrix
M(3,3)=0; M(1,1)=m; M(2,2)=m; M(3,3)=1/12*m*l^2;
for i=1:n+1;
    %Numerical solution
    q(i,3)=acos(cos(theta0)+v0*t(i)); q(i,1)=l/2*cos(q(i,3));
    q(i,2)=l/2*sin(q(i,3));
    D=[1,0,l/2*sin(q(i,3)); 0,1,-l/2*cos(q(i,3)); 1,0,-l/2*sin(q(i,3))];
    dq(i,:)=inv(D)*[0;0;v0];
    dD(1,:)=[0,0,l/2*cos(q(i,3))*dq(i,3)];
    dD(2,:)=[0,0,l/2*sin(q(i,3))*dq(i,3)];
    dD(3,:)=[0,0,-l/2*cos(q(i,3))*dq(i,3)];
    ddq(i,:)=inv(D)*[-dD(1:2,:)*dq(i,:)';-dD(3,:)*dq(i,:)'];
    h=[0;-m*g;0];
    lambda(i,:)=inv(D')*(M*ddq(i,:)'-h);
    %Analytical solution
    qa(i,3)=acos(cos(theta0)+v0*t(i)); qa(i,1)=l/2*cos(q(i,3));
    qa(i,2)=l/2*sin(q(i,3));
    dqa(i,3)=-v0/(l*sin(q(i,3)));
    ddqa(i,3)=-dqa(i,3)^2/tan(q(i,3));
    ddqa(i,1)=dqa(i,3)^2*l/2*cos(q(i,3))+ddqa(i,3)*l/2*sin(q(i,3));
    ddqa(i,2)=-dqa(i,3)^2*l/2*sin(q(i,3))+ddqa(i,3)*l/2*cos(q(i,3));
    ddqa(i,2)=-dqa(i,3)^2*l/2*sin(qa(i,3))+ddqa(i,3)*l/2*cos(qa(i,3));
    P(i)=-(1/12*m*l^2*ddqa(i,3)-m*ddqa(i,1)*l/2*sin(qa(i,3))+...
    m*ddqa(i,2)*l/2*cos(qa(i,3))+m*g*l/2*cos(qa(i,3)))/(l*sin(qa(i,3)));
    Ax(i)=m*ddqa(i,1)-P(i);
    By(i)=m*ddqa(i,2)+m*g;
end
```

3. Vibrations and rotor dynamics

3.1. Calculated example: Rotor dynamics

The rotor system shown in Figure 11 will be subject to analysis in the chapter.

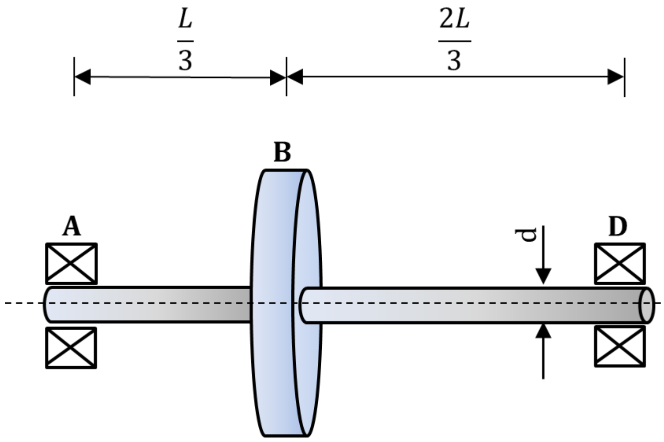


Figure 11 Offset rotor system

3.1.1. Analytical frequency calculation

Problem description: The mass of the rotor in the system shown in Figure 11 is *m=12 kg*, while the shaft is of length *L=0.8 m* with diameter *d=15 mm* and is made of steel with elastic modulus *E=210 GPa* and density $\rho = 7800\ kg/m^3$. The system operates at a service speed of *n=1000 rpm*. The shaft is supported in each end by a roller bearing with inner diameter $D_i = 18\ mm$, outer diameter $D_o = 24\ mm$ and 8 rolling elements.

Determine:

1. The frequency peaks, that would be enhanced in a frequency spectrum due to possible faults in the shaft-rotor system
2. The lowest whirling (bending) frequency of the rotor system
3. The log frequency (measuring rate) required for all possible bearing fault frequencies to be visible

Theory recap: This is the part, where it has turned out to be really hard to find a good, or at least decent, textbook to recommend. For now, we're stuck with the summaries contained in [4].

Solution: Frequencies due to faults in shaft-rotor system

The frequencies required, are the drive frequency proportional to the rpm-number along with it's 2nd and 3rd order harmonics. These are

$$f_{dr} = \frac{n}{60\ s} = \frac{1000}{60\ s} = 16.7\ Hz \qquad \text{(rotating imbalance)} \qquad 55.$$

$$2 \times f_{dr} = 33.3\ Hz \qquad \text{(misalignment)} \qquad 56.$$

$$3 \times f_{dr} = 50.0\ Hz \qquad \text{(bent/cracked shaft)} \qquad 57.$$

These would be the peaks enhanced in a frequency spectrum due to the specific faults listed. However, it is noted, that the drive frequency along with the 2nd and 3rd order harmonics usually are visible, and application of these for machine diagnostics and fault detection is a matter of interpretation, where frequency spectrums commonly are compared to spectrums based on measurements from healthy systems or historical data.

Solution: Whirling frequency – applying Dunkerley's formula

In order determine the whirling frequency, the eigenfrequencies for the rotor and for the shaft will be calculated separately and finally combined. While multiple methods are available for this, only Dunkerley's formula has this far been introduced, and we will for now stick to that. Our strategy, when considering the rotor separately, is to consider the rotor as a single DOF-system with spring stiffness calculated on basis of shaft stiffness, i.e. this is dependent on where on the shaft the rotor is mounted. In order to calculate the stiffness term, Bernoulli-Euler beam theory will be applied (see the standard solution to deflection problems in appendix A). We all know what that means … we better start out by calculating the cross-sectional constants A (area) and I (2nd order area moment of inertia) for the shaft:

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}(0.015\ m)^2 \quad = 1.77 \cdot 10^{-4}\ m^2 \qquad 58.$$

$$I = \frac{\pi}{64}d^4 = \frac{\pi}{64}(0.015\ m)^4 = 2.49 \cdot 10^{-9} m^4 \qquad 59.$$

The deflection for $x = L/3$ where the rotor is mounted is for a simply supported beam with offset concentrated load P in accordance with *appendix A, case c,* given by

$$\delta_B = \frac{a^2b^2}{3EIL}P \qquad a = \frac{L}{3}, b = \frac{2L}{3} \qquad 60.$$

This expression can be re-arranged to

$$P = \frac{3EIL}{a^2b^2}\delta_B = \frac{3EIL}{\left(\frac{L}{3}\right)^2\left(\frac{2L}{3}\right)^2}\delta_B \qquad 61.$$

The stiffness term is recognized as the proportionality factor and can now be calculated by

$$k_s = \frac{3EIL}{\left(\frac{L}{3}\right)^2\left(\frac{2L}{3}\right)^2} = \frac{3 \cdot 2.1 \cdot 10^9 N/m^2 \cdot 2.49 \cdot 10^{-9} m^4 \cdot 0.8m}{\left(\frac{0.8m}{3}\right)^2\left(\frac{2 \cdot 0.8m}{3}\right)^2} = 6.19 \cdot 10^4 \frac{N}{m} \qquad 62.$$

The circular eigen-frequency for the single DOF-system obtained considering solely the rotor is

$$\omega_n^{(rotor)} = \sqrt{\frac{k_s}{m}} = \sqrt{\frac{6.19 \cdot 10^4 \frac{N}{m}}{12\ kg}} = 71.8 \frac{rad}{s} \qquad 63.$$

Now that was the difficult part. Considering the shaft separately, this is contrary to the rotor, a *continuous system*, since mass and stiffness are not concentrated terms but have a spatial distribution. The circular eigen-frequency for a simply supported beam was introduced in [5] on analytical form

$$\omega_n^{(shaft)} = C\sqrt{\frac{EI}{SL^4}} \qquad S = A\rho \quad C = 9.87 \text{ (simply supported)}$$

$$= 9.87\sqrt{\frac{EI}{SL^4}} = 9.87\sqrt{\frac{2.1 \cdot 10^9 N/m^2 \cdot 2.49 \cdot 10^{-9} m^4}{7800 \frac{kg}{m^3} \cdot 1.77 \cdot 10^{-4}\ m^2 (0.8m)^4}} = 300 \frac{rad}{s} \qquad 64.$$

Dunkerley's formula can now be applied, in order to determine the total circular frequency of the system including both rotor and shaft

$$\frac{1}{(\omega_n)^2} = \frac{1}{\left(\omega_n^{(rotor)}\right)^2} + \frac{1}{\left(\omega_n^{(shaft)}\right)^2}$$

$$= \frac{1}{\left(71.8\frac{rad}{s}\right)^2} + \frac{1}{\left(300\frac{rad}{s}\right)^2} \qquad 65.$$

$$\rightarrow \omega_n = 69.8 \frac{rad}{s}$$

The corresponding natural frequency of the system, which can be observed in a frequency spectrum from the system, if this in stationary state is subjected to an impact test (hit with a hammer!), is given by

$$f_n = \frac{\omega_n}{2\pi} = 11.1 Hz \qquad 66.$$

It is noted that the lowest whirling frequency is lower than the drive frequency calculated in equation 55 meaning that the system operates *over-critically* when running at service speed. For poorly damped systems, this must be accounted for during run-up, since a drive frequency close to the whirling frequency will produce whirling with very large amplitudes, that are likely to cause damage to the system.

Solution: Bearing fault frequency log-rate

We know from the Nyquist criterium, that a measuring frequency of twice the frequency to be measured is required as minimum log. rate in order for frequencies to me visible in a frequency spectrum. As consequence, all bearing frequencies must be calculated, and we set the log rate to twice the maximum fault frequency. In order to do so, the diameters of the bearing cage and each rolling element is calculated as our first move

$$D = \frac{D_o + D_i}{2} = 21\ mm \qquad \text{(Cage diameter)} \qquad 67.$$

$$d = \frac{D_o - D_i}{2} = 3\ mm \qquad \text{(Rolling element diameter)}$$

The cage frequency is now calculated

$$f_c = \frac{\omega_c}{2\pi} = = \frac{1}{2} f_i \left(1 - \frac{d}{D}\right) = 7.1\ Hz \qquad \text{(Cage frequency – early wear indicator)} \qquad 68.$$

The frequencies due to defects and cracks on the inner race, outer race and rolling elements are

$$f_{bpfi} = \frac{N(f_i)\left(1 + \frac{d}{D}\right)}{2} = 76.2\ Hz \qquad \text{(Ball-pass frequency – inner race)} \qquad 69.$$

$$f_{bpfo} = \frac{N(f_i)\left(1 - \frac{d}{D}\right)}{2} = 57.1\ Hz \qquad \text{(Ball-pass frequency – outer race)} \qquad 70.$$

$$f_{bsf} = \frac{(f_i)}{2} \frac{D}{d} \left(1 - \left(\frac{d}{D}\right)^2\right) = 57.1\ Hz \qquad \text{(Ball-spin frequency)} \qquad 71.$$

$$f_{bpf} = 2 f_{bsf} = 114.3\ Hz \qquad \text{(Ball-pass frequency)}$$

The ball-pass frequency encountered due to a defect on a rolling elements passing the bearing races, is observed to be the maximum frequency obtained. The log rate is therefore

$$f_{log} = 2 f_{max} = 2 f_{bpf} = 228.6\ Hz \qquad \text{(Nyquist criteria - log. rate)} \qquad 72.$$

3.1.2. Calculation of whirling (bending) eigenfrequencies by FEA

The classical methods for calculation of whirling frequencies described in the previous section are often useful for smaller systems. However, for larger systems it turns out to be rather convenient to be able to use matrix-methods. Having done this already with all relevant theory summarized in [5], there's no reason to recap everything, since we already know that the undamped system will have an equation of motion on the form

$$[\mathbf{M}]\{\ddot{\boldsymbol{x}}\} + [\mathbf{K}]\{\boldsymbol{x}\} = 0 \qquad 73.$$

In order to find the eigen-frequencies (ω_n) and the corresponding eigen-modes $\{\boldsymbol{x}\}_m$, the following eigen-value problem must be solved

$$([\mathbf{K}] - (\omega_n)^2[\mathbf{M}])\{\boldsymbol{x}\}_m = 0 \qquad 74.$$

So we are going to need a stiffness- and a mass matrix for the rotor system in Figure 11. The discretization shown in Figure 12 will be applied. The system is split into three nodes each with a vertical translation and a rotation DOF, so each element-wise matrix is of size four by four (since the longitudinal displacement DOF is omitted). This yields two elements and a point mass where the rotor is mounted. The system is constrained by eliminating the translational displacement DOF's on node 1 and 3 (DOF's 1 and 5 in the unconstrained system) leaving us with a constrained system with four DOF's corresponding to the size of the global stiffness and mass matrices.

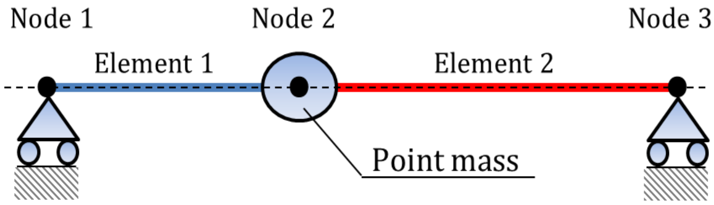


Figure 12 Offset rotor system, FEA idealization of the system shown in Figure 11

Due to our laziness ... or awesomeness (not quite sure), we can simply write two functions, `kElem` and `mElem` for generation of the element-wise stiffness- and mass matrices on basis of the element properties (see Appendix B). Having those available, the Matlab-code below generate and solve the eigenvalue problem. The lowest whirling frequency is determined to be *11.1 Hz* corresponding well to the value obtained analytically in equation 66.

```
clc; close all; clear all;
%Define system parameters
L=0.8;              %Shaft length
d=0.015;            %Shaft diam.
E=2.1*10^11;        %Module of elasticity
rho=7800;           %Density
m=12;               %Rotor mass
%Calculate cross-sectional constants for shafts
A=pi/4*d^2;         I=pi/64*d^4;
%Initialize K and M
K(6,6)=0;           M(6,6)=0;
%Add first element
K(1:4,1:4)=K(1:4,1:4)+kElem(E,I,L/3);
M(1:4,1:4)=M(1:4,1:4)+mElem(rho,A,L/3);
%Add second element
K(3:6,3:6)=K(3:6,3:6)+kElem(E,I,2*L/3);
M(3:6,3:6)=M(3:6,3:6)+mElem(rho,A,2*L/3);
%Add rotor mass
M(3,3)=M(3,3)+m;
%Implement constraints (displacement of node 1 and 3)
K(5,:)=[]; K(:,5)=[]; M(5,:)=[]; M(:,5)=[];
K(1,:)=[]; K(:,1)=[]; M(1,:)=[]; M(:,1)=[];
%Solve eigen-value problem
[Modes,EigVals]=eigs(K,M)
NatFrq=sqrt(EigVals(1,1))/(2*pi)
```

Since the mode shapes are now available in form of the eigen-vectors, we might as well plot those. The code below will do this based on a cubic interpolation function `CubeInt` (see Appendix B). The mode shapes are shown in Figure 13. These represent the physical forms by which the shaft with vibrate when passing the corresponding eigen-frequencies during run-up.

```
MNum=1;      %Plot mode shape
C1=CubeInt(0,0,Modes(1,MNum),L/3,Modes(2,MNum),Modes(3,MNum));
C2=CubeInt(L/3,Modes(2,MNum),Modes(3,MNum),L,0,Modes(4,MNum));
%Evaluate and plot the obtained solution over the length of the beam
n=100;
    for i=1:n+1
      x1(i)=(L/3)/n*(i-1);
      y1(i)=C1(1)*x1(i)^3+C2(2)*x1(i)^2+C1(3)*x1(i)+C1(4);
    end
    for i=1:n+1
      x2(i)=(2*L/3)/n*(i-1)+L/3
      y2(i)=C2(1)*x2(i)^3+C2(2)*x2(i)^2+C2(3)*x2(i)+C2(4);
    end
 figure; plot(x1,y1,'b-',x2,y2,'r');     %Plot deflections in new window
 xlabel('Length position along beam [m]')
 ylabel(['Mode Shape ' num2str(MNum)])
 grid on
```

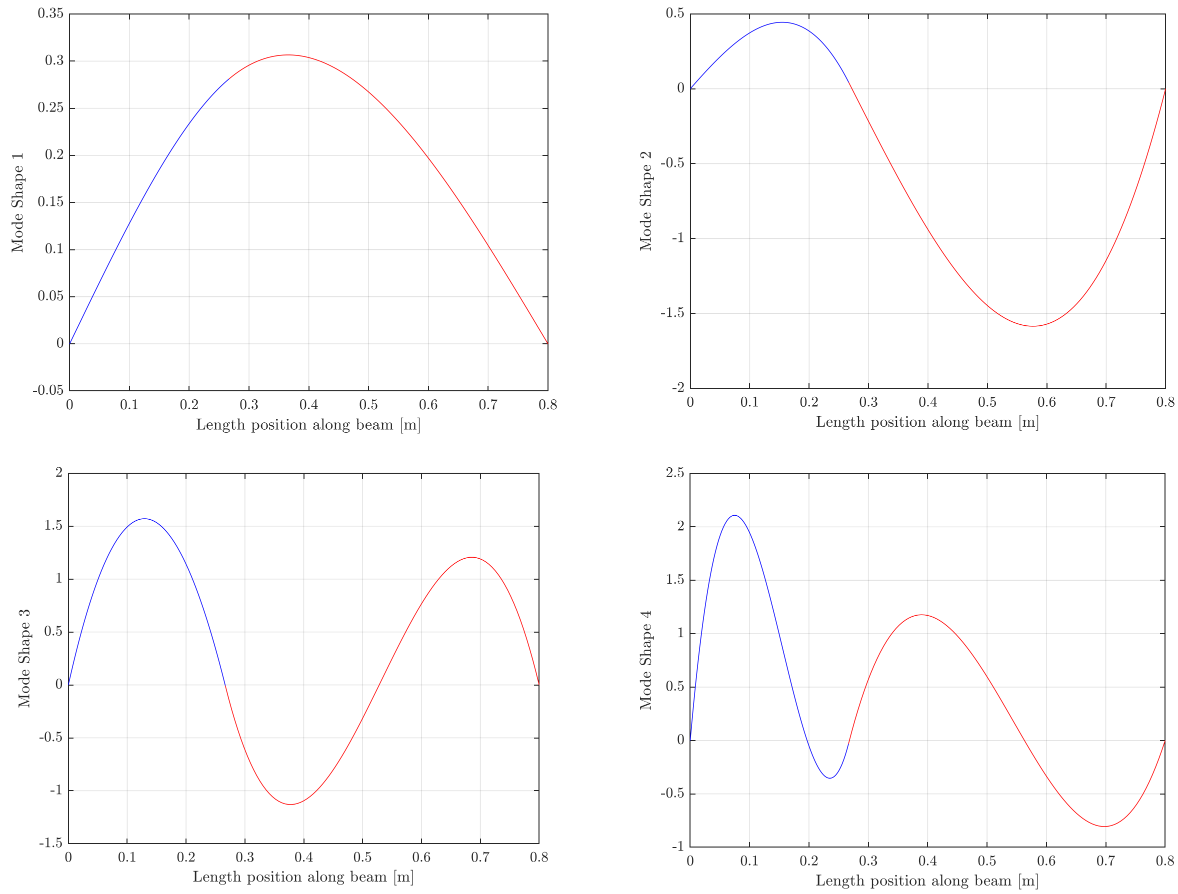
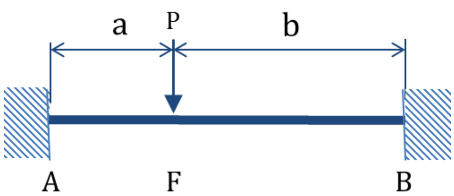
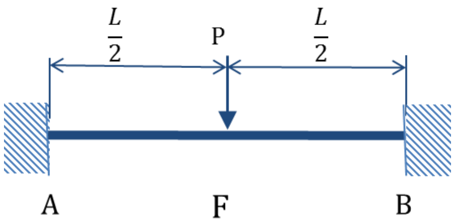
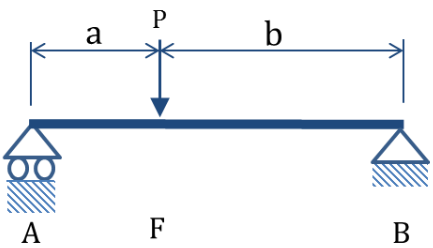
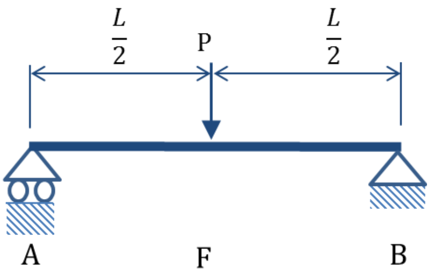
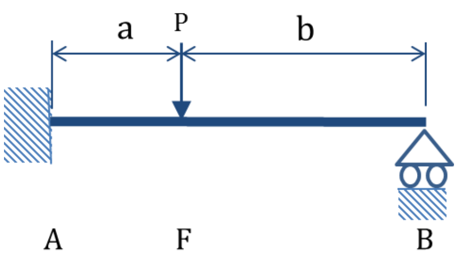
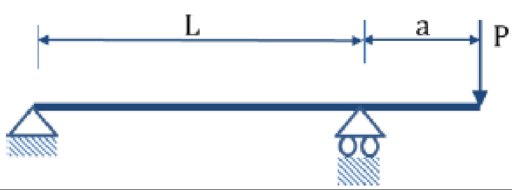


Figure 13 Mode shapes for the analyzed rotor system, see Figure 11

Appendix A: Max. defections for beam problems

a.		$\delta_F = \frac{Pa^3b^3}{3EIL^3}$
b.		$\delta_F = \frac{PL^3}{192EI}$
c.		$\delta_F = \frac{Pa^2b^2}{3EIL}$
d.		$\delta_F = \frac{Pl^3}{48EI}$
e.		$\delta_F = \frac{Pa^2b^3}{12EIL^3}(3L + a)$
f.		$\delta_S = \frac{PaL^2}{9\sqrt{3}EI}$ (max. deflection between supports) $\delta_F = \frac{Pa^2}{3EI}(L + a)$ (max. deflection – overhang)

Appendix B: Calculation of rotor-eigenfrequencies using FEA

```
function Output=kElem(E,I,L_elem);
    k_elem(1,1)=12*E*I/L_elem^3;  k_elem(1,2)=6*E*I/L_elem^2;
    k_elem(1,3)=-12*E*I/L_elem^3; k_elem(1,4)=6*E*I/L_elem^2;
    k_elem(2,1)=6*E*I/L_elem^2;   k_elem(2,2)=4*E*I/L_elem;
    k_elem(2,3)=-6*E*I/L_elem^2;  k_elem(2,4)=2*E*I/L_elem;
    k_elem(3,1)=-12*E*I/L_elem^3; k_elem(3,2)=-6*E*I/L_elem^2;
    k_elem(3,3)=12*E*I/L_elem^3;  k_elem(3,4)=-6*E*I/L_elem^2;
    k_elem(4,1)=6*E*I/L_elem^2;   k_elem(4,2)=2*E*I/L_elem;
    k_elem(4,3)=-6*E*I/L_elem^2;  k_elem(4,4)=4*E*I/L_elem;
    Output=k_elem;
```

```
function Output=mElem(rho,A,L_elem)
    m_Elem(1,1)=156;            m_Elem(1,2)=22*L_elem;
    m_Elem(1,3)=54;             m_Elem(1,4)=-13*L_elem;
    m_Elem(2,1)=m_Elem(1,2);    m_Elem(2,2)=4*L_elem^2;
    m_Elem(2,3)=13*L_elem;      m_Elem(2,4)=-3*L_elem^2;
    m_Elem(3,1)=54;             m_Elem(3,2)=m_Elem(2,3);
    m_Elem(3,3)=156;             m_Elem(3,4)=-22*L_elem;
    m_Elem(4,1)=m_Elem(1,4);     m_Elem(4,2)=m_Elem(2,4);
    m_Elem(4,3)=m_Elem(3,4);      m_Elem(4,4)=m_Elem(2,2);
    Output=rho*A*L_elem/420*m_Elem;
```

```
function Output=CubeInt(x1,y1,tht1,x2,y2,tht2)
    CoefMat(1:4,1:4)=0;     %Initialize array
    %Setup coefficient matrix and right-hand-side vector
    CoefMat(1,:)=[x1^3,x1^2,x1,1]; CoefMat(2,:)=[x2^3,x2^2,x2,1];
    CoefMat(3,:)=[3*x1^2,2*x1,1,0];CoefMat(4,:)=[3*x2^2,2*x2,1,0];
    B=[y1;y2;tht1;tht2];
    Output=CoefMat\B;
```

References

[1] F. Beer & R. Johnston, Dynamics, McGraw-Hill Education
(or the lecture slides from dynamics)
[2] P.E. Nikravesh, *Planar multibody dynamics*, 1st ed.: CRC Press, 2008.
[3] N.H. Østergaard, Introduction to computational forward dynamics with CarlaTheKraken, rev.01
[4] N.H. Østergaard, Lecture slides on condition monitoring
[5] N.H. Østergaard, Introduction to matrix methods in structural mechanics, rev. 01
[6] Shin & Hammond: fundamentals of signal processing for sound and vibration engineers