9. Instability of beam centrically loaded columns

Contents

Awesome people in Engineering mechanics, ch. 9

Leonhard Euler (1707-1783) was a Switch mathematician who is widely accepted in the tribal lore of the mathematicians for his work in number theory and complex algebra. However, he (for some unclear reason) also ended up inventing most of the beam theory along with the Bernuilli-Brothers while working in Sct. Petersburg. Thereby he has also earned immortality among engineers, who have named the theory for calculation of deflections in long and slender members after him. Furthermore, Euler was among the first to study linear buckling of beams. Though one could suspect that Euler mainly was interested in the indeed very fascinating bifurcation problems arising from the differential equations rather than developing engineering design tools, the obtained results are still used for engineering design calculations. His approach to elastic stability remained the only basis for understanding of structural instability until **Warner T. Koiter**s work on geometric non-linearities and initial post-buckling behavior in the mid-20th century

9.1 Introduction

Structures like beams, plates and shells may when subjected to compressive loads become unstable leading to buckling and possibly loss of load carrying ability leading to structural collapse. We will consider the simply supported end-loaded beam in Figure 9-2. If the beam is unstable, the equilibrium is not only fulfilled in the undeformed state, but also in a deformed configuration. Any small perturbation of the equilibrium state will then lead the structure to "snap" from the undeformed configuration to the deformed (since deformed states usually are of lower strain energy than undeformed). Instability of long and slender members like beam columns are of great practical importance, since loss of stability may occur for compressive stresses below the yield limit. Hence, stress and deformation based design as described in chapter 4, 6 and 8 will not be sufficient to ensure the structural integrity.

The present analysis will be limited to linear elastic beams subjected solely to centric compressive loads and fulfilling the Bernoulli-Euler assumptions, see chapter 8.

A typical example of a buckling of long and slender structures is thermal buckling of rail road tracks, see Figure 9-1, left.

Buckling of beams with significant internal bending moments is slightly more complex than the present theory and beyond our present scope. Most design norms and guidelines offer 'easy-to-use' design rules. However, the underlying theoretical approach is usually contained implicitly (i.e. well hidden).

Finally, before proceeding to the analysis, it should be noted, that initial imperfections reduce the load carrying ability. Hence, the obtained formulas are only valid for beams which are initially straight containing no significant imperfections.

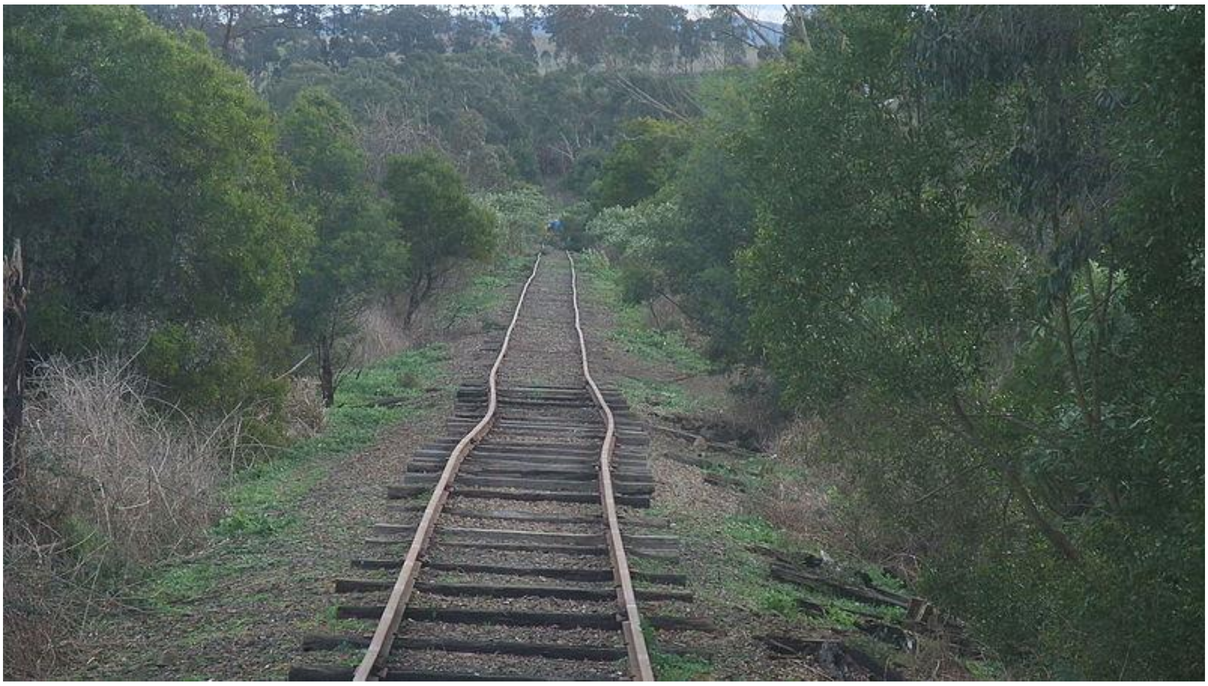


Figure 9-1 Thermal buckling of railway tracks

9.2 The Euler load

A simply supported beam subjected a compressive load will be considered, see Figure 9-2. The beam will be considered in the deformed state, and we will now consider for which loads the governing differential equation actually has a solution different from the trivial, i.e. undeformed configuration.

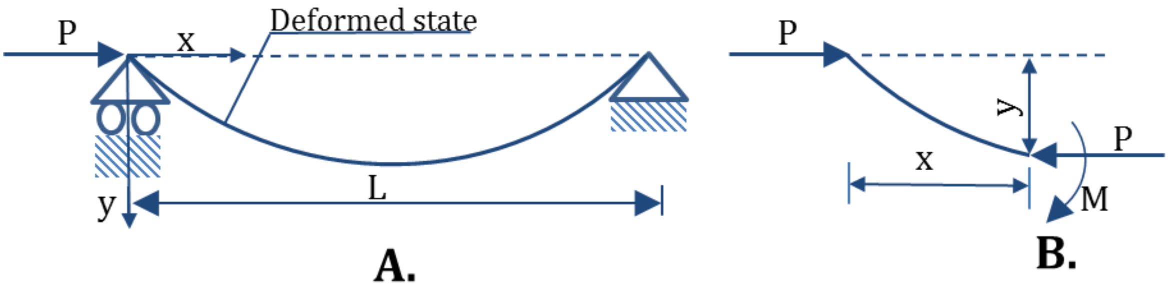


Figure 9-2 Simply supported beam subjected to compressive loads Please note the alternative coordinate system applied for this derivation

Considering Figure 9-2 B, the internal bending moment is given by

$$M(x) = -Py \tag{9-1}$$

in which P denotes the compressive load and y is the deflection as function of x. Rewriting *M(x)* in terms of u, we obtain

$$EI\frac{d^2u}{d^2x} + Py = 0$$
$$\rightarrow \frac{d^2u}{d^2x} + \frac{P}{EI}y = 0 \tag{9-2}$$

We have once again discover a second order homogenous differential equation (truly among the finest things in life). From mathematics, we know that this will have a solution on the form

$$y = A\sin\left(\sqrt{\frac{P}{EI}}x\right) + B\cos\left(\sqrt{\frac{P}{EI}}x\right) \tag{9-3}$$

(if you do not believe me, try differentiating twice in inserting the obtained expression in equation 9-2).

For the shown boundary conditions, we have

$$at\ x{=}0:\ \mathrm{y}(0) \rightarrow B = 0 \tag{9-4}$$

Equivalently, we have

$$at\ x{=}L:\ \mathrm{y(L)=0} \rightarrow 0 = A\sin\left(\sqrt{\frac{P}{EI}}L\right) \tag{9-5}$$

Except for the trivial solution, A=0, this equation can only be fulfilled if

$$\sin\left(\sqrt{\frac{P}{EI}}L\right) = 0 \tag{9-6}$$

which is the case if

$$\sqrt{\frac{P}{EI}}L = n\pi \tag{9-7}$$

The lowest value of P, for which instability may occur is obtained by setting *n=1*. This value is called *the critical load* and is denoted P_{cr} (often also referred to using the term *Euler load* P_E) and is on basis of the equation above by calculus obtained as

$$P_{cr} = \frac{\pi^2 EI}{L^2} \tag{9-8}$$

This value of compressive load is not to be exceeded if the load carrying ability is not to be compromised.

9.3 Influence of boundary conditions: the Euler length

The question is now how to adapt the obtained expression to boundary conditions different from the ones applied in the analysis above (simply supported). Solving the obtained differential equation for different boundary conditions, it turns out that the critical load can be written on the general form

$$P_{cr} = \frac{\pi^2 EI}{{L_E}^2} \tag{9-9}$$

in which L_E is the Euler length. This can for the most common boundary conditions for practical purposes be obtained by looking it up in tables, see for example Table 1.

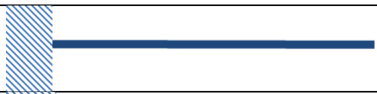
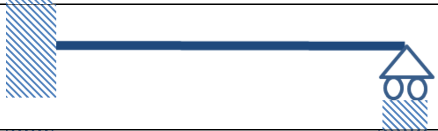
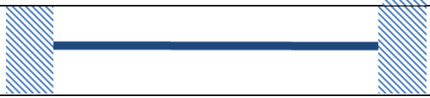
Case	**Boundary conditions**	**Euler length L_E**
Simply supported		1.0L
Cantilever		2L
Simply supported cantilever		0.699L
Double cantilever		0.5L

Table 1 The Euler length for different boundary conditions

The load a long and slender beam can carry, is highly dependent on imperfections in the initial state, see Figure 9-3. If the beam is slightly crooked before loaded, it will buckling by deflecting along a curved equilibrium path after a limit point has been passed. Furthermore, non-centric loads causing a moment may also cause buckling, but these are usually accounted for by calculating deformations rather than a critical load causing bifurcation. These phenomena require further analysis which is beyond the scope of a basic strength of materials course[1]

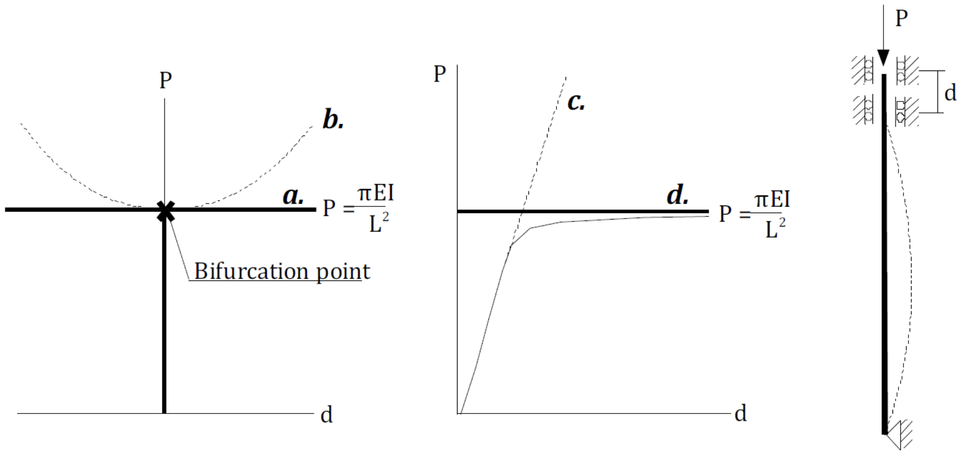
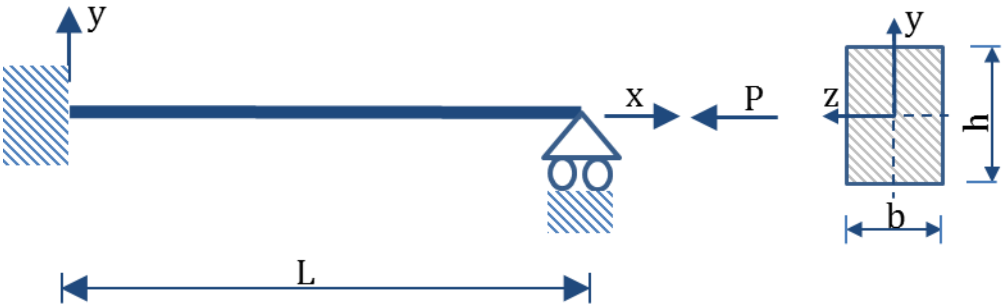


Figure 9-3 Load-displacement path for a. straight beam with small deflections and rotations, b. straight beam with small deflections and large rotations, c. imperfect beam with large deflections and rotations, primary unstable solutions, d. imperfect beam with large deflections and rotations, stable equilibrium path corresponding to buckling

[1] Lecture notes about buckling from MIT Courseware available here, here and here

Calculated example 9A: Design of rectangular beam against compressive loads



A rectangular beam with h=20 mm, b=10 mm and L=0.5 m is made of steel with elastic modulus E=210 GPa and max. allowable design stress σ_Y=200 N/mm². The shown boundary conditions applies both in the xy and xy-plane. Calculate the maximum compressive load that can be carried by the beam.

Solution:

Design against instability:

Noticing that bending around the weak axis (the y-axis) governs the design, the critical load is

$$P_{cr} = \frac{\pi^2 EI}{{L_E}^2} = \frac{\pi^2 210000 \frac{N}{mm^2}\left(\frac{1}{12} 20mm(10mm)^3\right)}{(0.699 \cdot 500mm)^2} = 28280 \text{ N}$$

Design against compressive stresses:

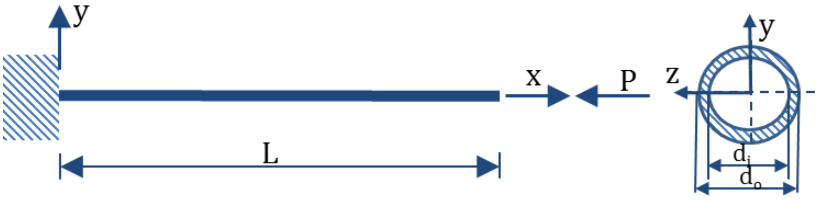
The maximum allowable compressive stress is given by

$$P_C = \sigma_Y A = 200 \frac{N}{mm^2}(10mm \cdot 20mm) = 40000\ N$$

Since $P_{cr}<P_C$, the maximum compressive load the column can carry is $P_{cr}=28280\ N$

Problems

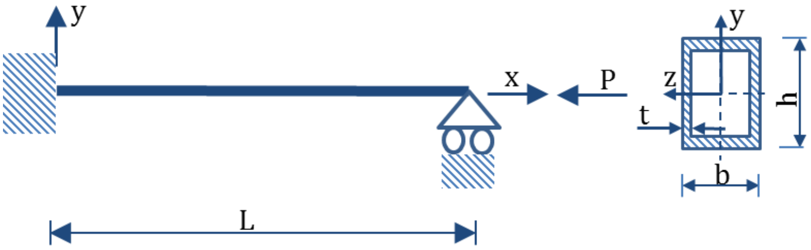
Problem 9.1



For the simply supported beam-column shown, the geometry is given by the parameters L=6 m and diameters d_o=220 mm and d_i=200 mm. The beam is made of steel with a module of elasticity E=210 GPa and yield strength σ_y=355 N/mm². Calculate the maximum axial compressive load P that can be carried by the beam column

Ans: P_{cr}=524 kN (P_c=2342 kN)

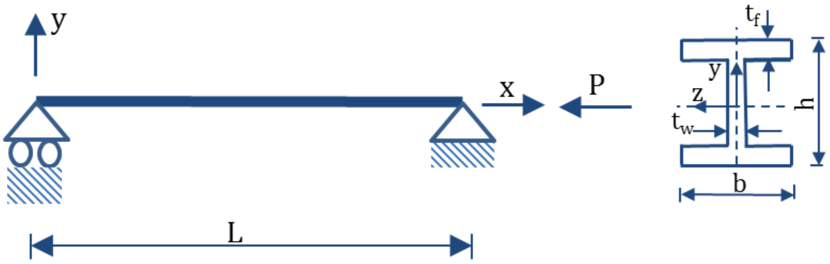
Problem 9.2



The aluminum beam column shown is of length L=8 m and has a module of elasticity E=69 GPa and yield stress σ_Y=55 N/mm². The geometry of the cross-section is given by h=200 mm, b=120 mm and t=12 mm. The boundary conditions shown on the figure above applies both in the xy and xz plane. Calculate the maximum compressive load P that can be carried by the beam column

Ans: P_{cr}=345 kN (P_c=391 kN)

Problem 9.3



The steel beam shown in the figure above is of length L=7.5 m and has a module of elasticity E=210 GPa and yield stress σ_Y=235 N/mm². The geometry of the cross-section is given by h=325 mm, b=310 mm, t_w=15 mm and t_f=25 mm. The boundary conditions shown on the figure above applies both in the xy and xz plane. Calculate the maximum compressive load P that can be carried by the beam column

Ans: P_{cr}=4577 kN (P_c=4612 kN)

Problem 9.4

Solve problem 2.5 and determine by calculation if the temperature increase will cause the bar to buckle sideways.

Ans; P_{cr}=67468 N