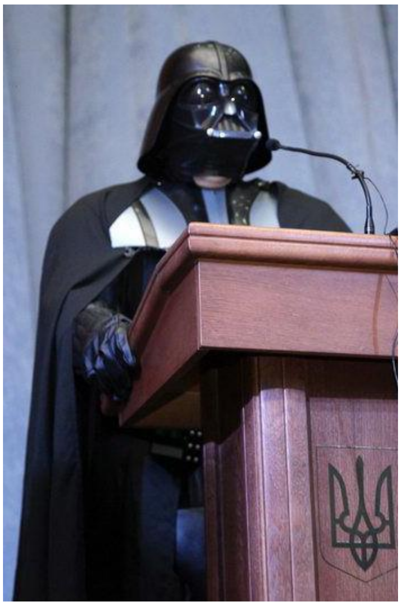
8. Deflection of beams

Contents

Nomenclature

M	Bending moment [Nmm]	A	Area
V	Shear force [N]	E	Module of elasticity [N/mm^2]
q,w	Distributed load [N/mm]	κ	Curvature [1/mm]
x	Longitudinal beam coordinate [mm]	R	Bending radius [mm]
u	Deflection [mm]	s	Arc length [mm]
θ	Rotation [rad]	P	Load [N]
C	Integration constants	I	Second order area moment of inertia [mm^4]



Awesome people in Engineering mechanics, ch. 8

Stephen Timoshenko (1878 –1972) was a Ukanian born Engineer, who by many is considered the father of the modern civil engineering and additionally the dark lord of structural mechanics. Having moved back and fort between the technical universities in Sct. Petersburg, Kiev and Zagreb in the politically turbulent early 20th century, he immigrated to the US in 1920 and soon became a professor at the university of Michigan. From 1936 to 1970 he was a professor for structural mechanics at Standford University. When retiring in 1970, he moved to Wuppertal, West Germany, to be with his daughter, where he remained until his death. Timoshenko published a wide range of textbooks on structural mechanics which remain highly recognized. Among those are 'Mechanics of Materials' (today with James Geere as main author). Timoshenkos most known contribution to engineering sciences is the short beam- or Timoshenko beam theory applied for analysis of members which have length to height ratio below 12. Though this theory is not presented in the present notes, it remain of great importance in advanced structural mechanics.

(Photos from Wikimedia)

8.1 Introduction

It has earlier been observed that symmetric beams subjected to force-couples (bending moments) in each end will deform in a manner so the centroid axis forms a segment of a circular arc with constant curvature and bending radius related by $\kappa=1/R$. For prismatic beams subjected to general loadings, the deformation shape of the centroid axis is more complex. In the present chapter, we have as scope to derive a theoretical framework that can be applied for calculation of the geometry of the centroid axis in the deformed state by application of plane geometry. This curve will be denoted 'the elastic curve'.

Though the flexure formula for calculation of bending stresses was derived for beams in pure bending, it still provides results with acceptable accuracy for beams subjected to general loading, if the bending moment applied in the formula is calculated based on the actual equilibrium.

The entire theoretical framework for analysis of beam systems is usually called the 'Bernuilli-Euler beam theory' or 'the technical theory of beams'. The Bernuilli-Euler beam theory constitutes one of a very few classical engineering tools to which Sir Isaac Newton (the greatest engineer of all times) did not provide any significant contributions. The theory was mainly derived by the Switch mathematician Leonard Euler (1707-1783) who is also to be considered among the founding fathers of the mechanical engineering discipline. The present section should make it clear to students why mechanics of materials along with dynamics were considered a mathematical discipline until the early 20th century. The earlier mentioned 'short beam theory', which is not part of the present scope, is a modern extension of the theory valid for beams with $L/h < 12...15$ derived by Ukrainian-American civil engineer Stephen Timoshenko 1878-1972, who by many is considered the father of modern engineering technics.

It is of great importance to note that the theory described in these notes is valid only for beams with cross-sections containing at least one axis of symmetry. If this is not the case, the expressions required to calculate both defections and stresses are complicated significantly. The mathematical expressions for the deformed shape of the beam centroid-axis is derived on basis of bending moments obtained assuming that plane cross sections remain plane. This is often referred to as 'the Bernuilli-Euler assumption' and is in general valid for beams with a length to height ratio $L/h > 12...15$, which is a rule of thumb for when the theory provides accurate results. Hence, the analysis is limited to considerations of long and slender members.

The material of the beam must be linear elastic, isotropic (meaning, that the material parameters are directionally independent) and homogeneous. Hence, the theory is only valid as long as the stresses are within the elastic range.

The equations applied for calculation of deflections, curvature, internal forces and moments are valid for small deformations and rotations, since a first order Taylor approximation is used to approximate the quantities throughout a small beam segment.

8.2 Derivations of the governing differential equations

A beam subjected to general loadings will be considered, see Figure 8-1. We will focus our efforts on deriving an expression for the deflection of the neutral axis, which as mentioned earlier will be denoted 'the elastic curve' in the deformed state. It is important to note that the curvature is no longer constant but a function of arclength s.

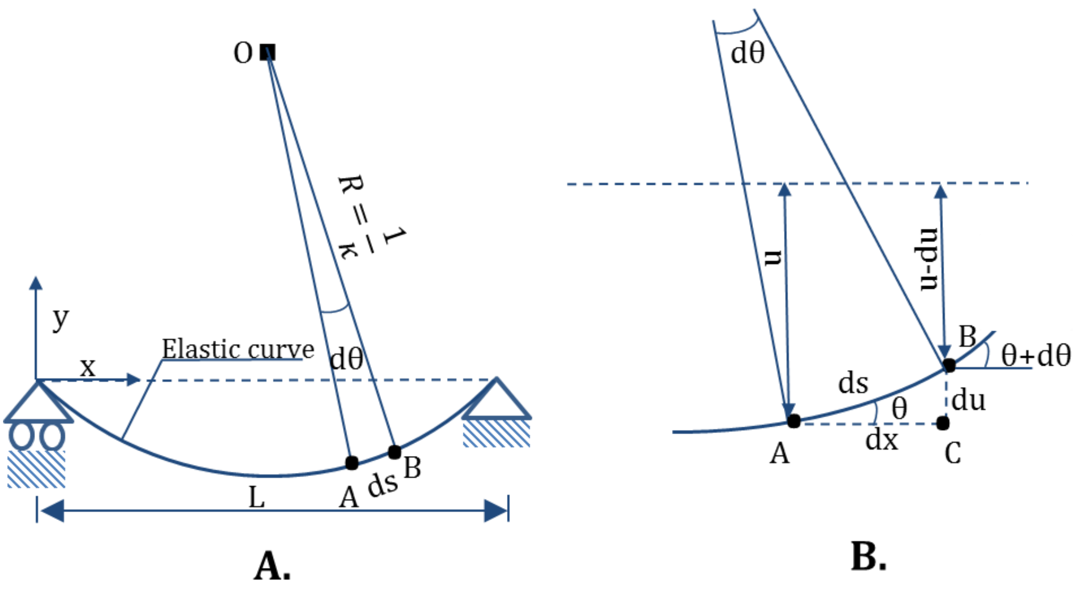


Figure 8-1 The elastic curve (deformed centroid axis) of a beam in general loadings

Initially, the right angled triangle ABC will be considered. We derive

$$\begin{aligned} &\Delta\text{ABC: } \cos\theta = \frac{dx}{ds} \\ &\rightarrow ds\cos\theta = dx \\ &\rightarrow ds \approx dx \qquad \text{since } \cos\theta \approx \theta \text{ for } \theta \ll 1 \end{aligned} \tag{8-1}$$

Hence, we may differentiate with respect to x rather than s as long as strains and deflections are small. This simplifies matters significantly. Furthermore, we have

$$\begin{aligned} &\Delta\text{ABC: } \sin\theta = \frac{du}{ds} \approx \frac{du}{dx} \\ &\rightarrow \theta \approx \frac{du}{dx} \qquad \text{since } \sin\theta \approx \theta \text{ for } \theta \ll 1 \end{aligned} \tag{8-2}$$

Now considering the right angled triangle OAB, the following expression is derived

$$\begin{aligned} &\Delta\text{OAB: } \sin(d\theta) = \frac{ds}{R} \approx \frac{dx}{R} \\ &\rightarrow d\theta \approx \frac{dx}{R} \\ &\rightarrow \frac{d\theta}{dx} \approx \frac{1}{R} = \kappa \end{aligned} \tag{8-3}$$

This relation can alternatively be derived based on the relation for the arc length of the segment *ds = Rdθ*. However, we recall from our analysis of beams in pure bending in chapter 4, that the curvature is related to the bending moment by

$$\kappa = \frac{M}{EI} \qquad \text{for } \theta \ll 1 \tag{8-4}$$

Combining the two equations above, we obtain

$$\frac{d\theta}{dx} \approx \frac{M}{EI} \tag{8-5}$$

The expressions applied in this context are only valid for small deflections. If this assumption is violated, the curvature is no longer given by the equation above, but by the full expression known from differential geometry. This is given by

$$\kappa = \frac{d\theta}{ds} = \frac{\frac{d^2u}{dx^2}}{\sqrt{\left(1+\left(\frac{du}{dx}\right)^2\right)^3}} \approx \frac{d^2u}{dx^2} \qquad \text{for } \theta \ll 1, \text{ since } \frac{du}{dx} \ll 1 \tag{8-6}$$

The full expression required for large deflections in the finite strain theory would complicate our analysis significantly. Hence, the simplified expressions are extremely convenient.

8.3 The Bernuilli-Euler beam theory

The complete framework available for analysis of long and slender beams can now be summarized in form of the following four coupled differential equations to be solved wrt. appropriate boundary conditions

$$\frac{du}{dx} = \theta(x) \qquad (8\text{-}7) \qquad\qquad \frac{dM}{dx} = V(x) \qquad (8\text{-}8)$$

$$\frac{d\theta}{dx} = \frac{M(x)}{EI} \qquad (8\text{-}9) \qquad\qquad \frac{dV}{dx} = -w(x) \qquad (8\text{-}10)$$

Calculated example 8A: Deflection of cantilever beam

For the cantilever beam shown in Figure 8-2, determine the equation of the elastic curve by direct integration.

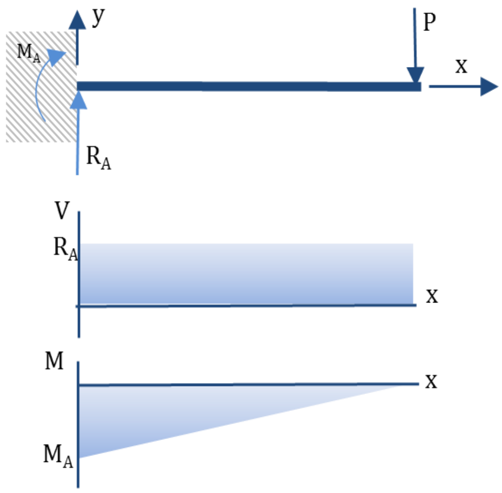


Figure 8-2

Solution

Based on the free body diagram shown in Figure 8-2, the solution to the moment and vertical force equilibrium equations is obtained as R_A=P and M_A=-PL. Hence, the internal sectional force in the beam is constant throughout the entire length. We have

$V = P$

The equation of the moment curve is obtained by

$M = \int P\,dx = Px + C_1$

The integration constant C_1 is obtained on basis of the boundary condition

$M(x = L) = 0$

$\rightarrow P \cdot L + C_1 = 0$

$\rightarrow C_1 = -PL = M_{max}$

The elastic curve can now be obtained as the solution to the following differential equation

$$\frac{d^2u}{dx^2} = \frac{M}{EI} \rightarrow \iint \frac{d^2u}{dx^2} dxdx = \frac{1}{EI}\iint M\,dxdx$$

$$= \frac{1}{EI}\iint Px - PL\,dxdx = \frac{1}{EI}\int \frac{P}{2}x^2 - PLx + C_2\,dx$$

$$= \frac{1}{EI}\left(\frac{P}{6}x^3 - \frac{PL}{2}x^2 + C_2 x\right) + C_3$$

This is the equation for the elastic curve. The boundary conditions are applied in order to calculate the integration constants

$$\theta(x = 0) = \frac{du}{dx}(x = 0) = 0$$

$$\frac{1}{EI}\left(\frac{P}{2} \cdot 0^2 - PL \cdot 0 + C_2\right) = 0$$

$\rightarrow C_2 = 0$

$u(x = 0) = 0$

$$\rightarrow \frac{1}{EI}\left(\frac{P}{6} \cdot 0^3 - \frac{PL}{2} \cdot 0^2\right) + C_3 = 0, \rightarrow C_3 = 0$$

8.4 Standard solutions

In table 1, standard solutions for deflection and rotation for beams with given boundary conditions are presented. This can be applied directly for design.

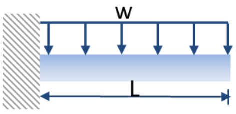
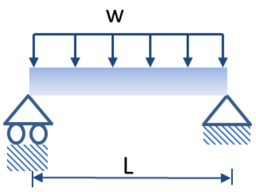
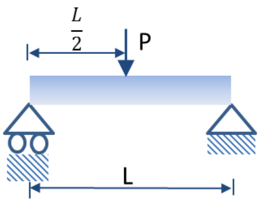
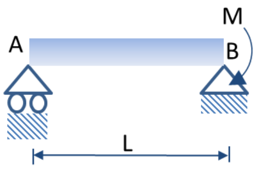
Load case and BCDs	Max. Deflection	End slope	Equation of elastic curve
	$-\dfrac{PL^3}{3EI}$	$-\dfrac{PL^2}{2EI}$	$u = \dfrac{P}{6EI}(x^3 - 3Lx^2)$
	$-\dfrac{wL^4}{8EI}$	$-\dfrac{wL^3}{6EI}$	$u = -\dfrac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$
	$-\dfrac{ML^2}{2EI}$	$-\dfrac{ML}{EI}$	$u = -\dfrac{M}{2EI}x^2$
	$-\dfrac{5wL^4}{384EI}$	$\pm\dfrac{wL^3}{24EI}$	$u = -\dfrac{w}{24EI}(x^4 - 2Lx^3 + L^3x)$
	$-\dfrac{PL^3}{48EI}$	$\pm\dfrac{PL^2}{16EI}$	For $x \leq \frac{L}{2}$: $u = \dfrac{P}{48EI}(4x^3 - 3L^2x)$
	$\dfrac{ML^2}{9\sqrt{3}EI}$	$\theta_A = +\dfrac{ML}{6EI}$ $\theta_B = -\dfrac{ML}{3EI}$	$u = -\dfrac{M}{6EIL}(x^3 - L^2x)$

Table 8-1 Standard solutions to beam problems

8.5 The principle of superposition for deflection of beams

The principle of superposition also applies for rotation and deflection of beams in exactly the same fashion as for bending moment and shear force diagrams. Hence, the standard solutions given in Table 1 can be added by superposition to form more complex load cases.
It will be demonstrated how this works in the following calculated example. It is considered most efficient to explain the principle of superposition directly on basis of two calculated examples.

Calculated example 8B: Maximum deflection calculated by superposition

For the beam shown in Figure 8-3A, calculate the elastic curve and the maximum deflection.

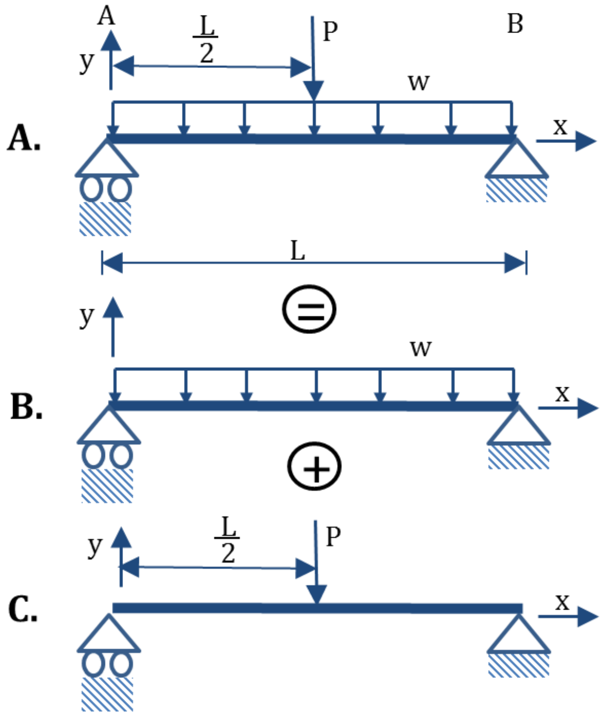


Figure 8-3

Solution

Reviewing the problem, it is realized that the problem can be separated into **a)** a simply supported beam subjected to a distributed load and **b)** a simply supported beam subjected to a concentrated load. By superposition, the elastic curve can be obtained as the sum of the elastic curves for the two separate cases shown in Figure 8-3B and C.

Using the standard solution in Table 8-1, the maximum deflection is obtained for x=L/2 and is given by

$$u_{max} = u_P\left(x = \frac{L}{2}\right) + u_w\left(x = \frac{L}{2}\right) = -\frac{5wL^4}{384EI} - \frac{PL^3}{48EI}$$

The equation for the elastic curve is for x<L obtained in a similar fashion

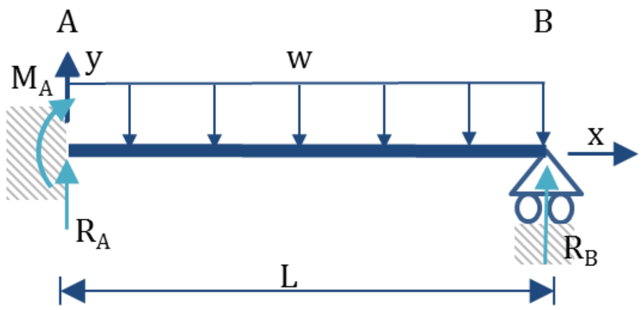
$$u = \left(-\frac{w}{24EI}(x^4 - 2Lx^3 + L^3x)\right) + \left(-\frac{P}{48EI}(4x^3 - 3L^2x)\right)$$

8.6 Statically indeterminate beams

It is recalled that the term 'statically indeterminate' refers to a beam configuration where the number of equilibrium equations available are insufficient in order to determine the reactions. In this section it will be demonstrated how statically indeterminate beam problems can be solved by including the differential equations for rotation and deflection in the formulation of the problem.

8.6.1 Solution by direct integration

Calculated example 8C: The force equilibrium of an end-supported cantilever solved by direct integration



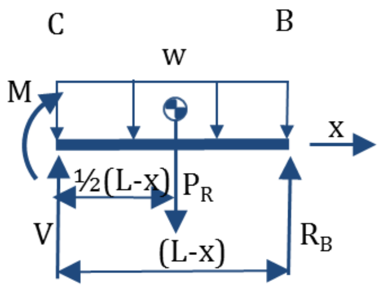


Figure 8-4 **Figure 8-5**

For the beam shown in Figure 8-4, determine the vertical reaction force in point B.

Solution

By counting the reactions, it is realized that only two equations of equilibrium are available for calculation of three unknowns (R_A, R_B and M_A), so the beam problem is statically indeterminate. In order to obtain a solution to the static equilibrium, the equations for the rotation and deflection are required. By adding a section (a cut!) between point A and B and considering the right segment, the free-body diagram in Figure 8-5 is obtained. The equation of the internal bending moment curve is now given by

$$M + w(L-x)\cdot\frac{L-x}{2} - R_B(L-x) = 0 \rightarrow M = -w(L-x)\cdot\frac{L-x}{2} + R_B(L-x)$$

$$= -\frac{w}{2}(L^2 + x^2 - 2Lx) + R_B L - R_B x$$

The deflection is now obtained by integration

$$u = \iint M\,dx = \iint -\frac{w}{2}(L^2 + x^2 - 2Lx) + R_B L - R_B x\,dx$$

$$= -\frac{w}{2}\left(\frac{L^2}{2}x^2 + \frac{1}{12}x^4 - \frac{L}{3}x^3\right) + \frac{R_B L}{2}x^2 - \frac{R_B}{6}x^3 + C_1 x + C_2$$

The integration constants are determined on basis of the boundary conditions

$\theta(x=0) = 0 \rightarrow C_1 = 0$ u$(x=0) = 0 \rightarrow C_2 = 0$

On basis of the third boundary condition, a third equation is now obtained

$u(x=L) = 0$

$$\rightarrow -\frac{w}{2}\left(\frac{L^2}{2}L^2 + \frac{1}{12}L^4 - \frac{L}{3}L^3\right) + \frac{R_B L}{2}L^2 - \frac{R_B}{6}L^3 = 0$$

$$\rightarrow -w\left(\frac{L^4}{4} + \frac{L^4}{24} - \frac{L^4}{6}\right) + \frac{R_B L^3}{2} - \frac{R_B L^3}{6} = 0$$

$$\rightarrow -\frac{3wL^4}{24} + \frac{2R_B L^3}{6} = 0 \rightarrow \frac{R_B L^3}{3} = \frac{3wL^4}{24}$$

$$\rightarrow R_B = \frac{9}{24}wL = \frac{3}{8}wL$$

The reactions in point A can now be obtained on basis of conventional force and moment equilibrium.

8.6.2 Solution by superposition

Calculated example 8D: The force equilibrium of an end-supported cantilever solved by superposition

Force the beam shown in Figure 8-6, determine the vertical reaction in point B applying the principle of superposition.

Solution

Decomposing the problem as shown in Figure 8-6 and the standard solution contained in Table 8-1, the boundary condition in point B enables us to obtain a single equation, which can be solved for R_B:

$u(x = L) = 0$

$u(x = L) = u_P(x = L) + u_w(x = L)$

$= -\frac{wL^4}{8EI} + \frac{PL^3}{3EI}$

Setting $P=R_B$, the following equation is obtained

$u(x = L) = 0 = -\frac{wL^4}{8EI} + \frac{R_B L^3}{3EI}$

$\rightarrow R_B = \frac{3}{8} wL$

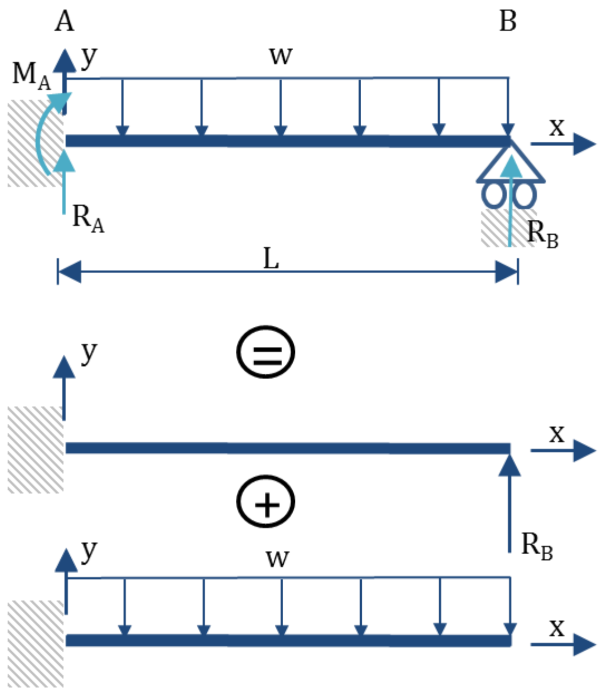


Figure 8-6

Problems

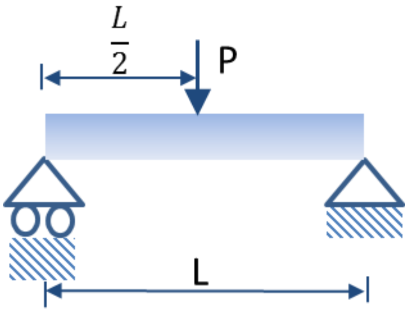


Figure P8.1

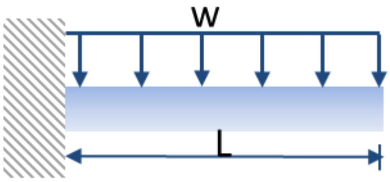


Figure P8.2

Problem 8.1: For the beam shown in Figure P8.1, determine a) the equation of the elastic curve, b) the maximum deflection, c) the maximum angle of rotation
Problem 8.2: For the beam shown in Figure P8.2, determine a) the equation of the elastic curve, b) the maximum deflection, c) the maximum angle of rotation

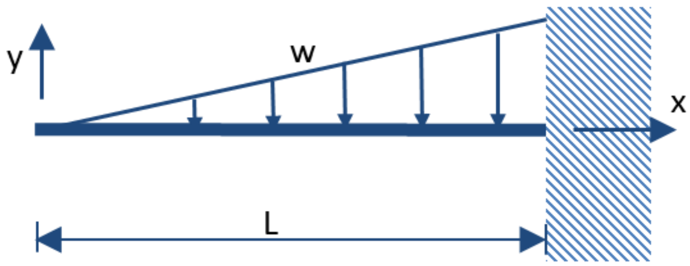


Figure P8.3
Problem P8.3: For the beam shown in Figure P8.3, determine a) the equation of the elastic curve, b) the maximum deflection, c) the maximum angle of rotation

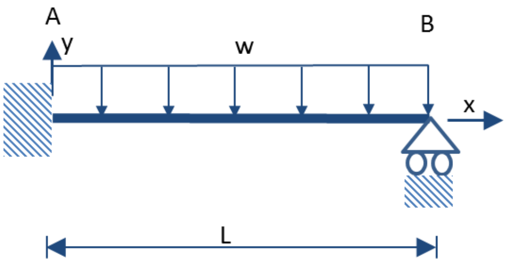


Figure P8.4

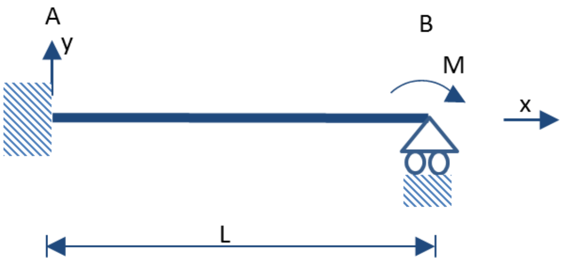


Figure P8.5

Problem 8.4:
For the beam shown in Figure P8.4, calculate the reaction in B in terms of w and L
(Hint: you may apply the standard solutions in table 8.1)

Problem 8.5:
For the beam shown in Figure P8.5, calculate the reaction in B in terms of M and L
(Hint: you may apply the standard solutions in table 8.1)

Problem 8.6:
For the beam shown in Figure P8.6, calculate the maximum deflection.
The system parameters are given by di=200 mm, do=220 mm, L=3 m, w=5 kN/m and P=9 kN.
The beam is made of steal with elastic modulus E=210 GPa.
(Hint: you may apply the standard solutions in table 8.1)
Ans: u=-17.20 mm (downwards)

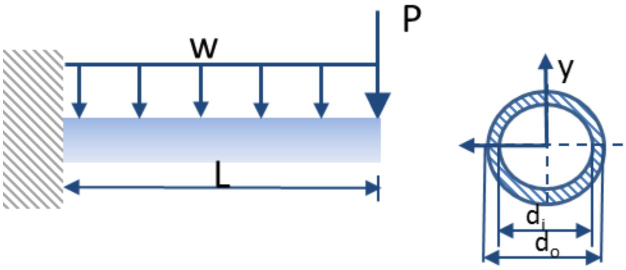


Figure P8.6