6. Beam Design

Contents

Nomenclature

[Nothing new in this chapter]

6.1. Introduction

In the present chapter, the methods for stress-based design of prismatic beams will be finalized. Initially, the framework for calculation of sectional forces based on the two coupled first order differential equations in M and V will be formalized. Afterwards, we will on basis of calculated examples demonstrate how the developed framework is applied in order to solve design problems.

The presented theory is valid for long and slender prismatic beams of linear elastic homogenous materials which are subjected to general loadings only causing states of deformations characterized by small deflections and rotations.

Figure 6-1 Example of a structure dominated by beams as structural elements – a railway bridge (photo by author)

6.2. Equilibrium of internal and external forces

Our calculation of stresses will as always be based on internal (or sectional) forces and moments. Hence, we will start out with a summary of the basic knowledge from statics on how to determine sectional forces and moments in beams. Considering a beam segment of infinitesimal length in Figure 6-2, sectional forces and moments on the right and left boundary will be considered positive in the shown directions. It is of great importance to note that we now operate with two different coordinates systems:

- A force coordinate system applied when writing equilibrium equations - here you may sum up forces and moments with positive directions of your own choice (as long as you apply them consistently).
- A material coordinate system, which is applied when drawing/calculating the distribution of sectional forces and moments. In this coordinate system the positive directions for internal forces and moments are shown in Figure 6-2. If applying the convention consistently, addition of an external load will cause an increase in the sectional force diagram of same magnitude and direction as the applied force (the same applies for moments).

In order to relate the internal force quantities on the left face to the quantities on the right face, these will (by an in structural mechanics commonly used trick) be developed by a first order Taylor approximation. We recall that Taylors formula for a first order approximation is given by

$$f(x) \approx f(a) + f'(a)(x-a)^n \tag{6-1}$$

Applying this to express any force quantity on the right face in terms of the quantity of the left face along with the corresponding derivative in x, we obtain

$$F(dx) \approx F + \frac{dF}{dx}(dx) = F + dF \tag{6-2}$$

This will be utilized in the following. The approximated quantities are shown in Figure 6-1.

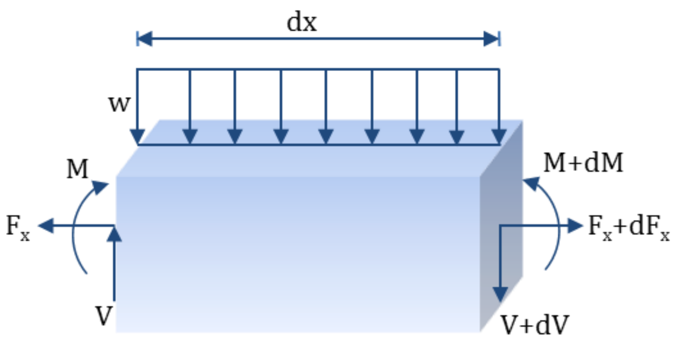


Figure 6-2 A beam segment subjected to a distributed load – note that the sign convention for internal forces (material coordinates) is defined in this figure

Initially, when formulating the equilibrium we notice that this is done in force coordinates. Hence, we apply the sign conventions and principles we know from statics. Considering the equilibrium of forces in the vertical direction we have

$$\begin{aligned} &\sum F_y = 0 \\ &\rightarrow V - qdx - (V + dV) = 0 \\ &\rightarrow \frac{dV}{dx} = -q \end{aligned} \tag{6-3}$$

The moment equilibrium around the left entity of the beam segment yields

$$\begin{aligned} &\sum M_z = 0 \\ &\rightarrow M - (M + dM) + Vdx + qdx\frac{dx}{2} = 0 \\ &\rightarrow \frac{dM}{dx} = V \end{aligned} \tag{6-4}$$

in which we have applied $dx^2 \approx 0$ equivalently to stating in words that the product of two very small numbers is so insignificantly small, that it can be neglected without loss of accuracy (we say that higher order terms are neglected).
These two equations are very central in beam theory, since they prescribe the distribution of internal force and moment as functions of the length coordinate x.
The following strategy can be applied for design of beams of statically determinate beam structures

Problem solving strategy

1. Use static equilibrium ($\sum F_x = 0, \sum F_y = 0, \sum M_z = 0$) to calculate all reactions acting on the beam
2. On basis of the principles from statics, construct the internal shear force and bending moment diagrams and find the maximum values, V_{max} and M_{max}.
3. Apply the Flexure Formula and Grasshof's formula to relate stresses and internal forces in terms of cross-sectional parameters.

In the following, we will demonstrate how to apply the derived theoretical framework in conjunction with stress calculation for practical design problems.

Calculated example 6A: Beam subjected to distributed load

The shown wooden beam has quadratic cross-section and is of length L=3m. The beam is subjected to a distributed load w=1.5 kN/m. If the maximum allowable normal stress is given by σ_{all}=15 N/mm² and the maximum allowable shear stress is given by τ_{all}=2.5 N/mm², determine the height h by calculation.

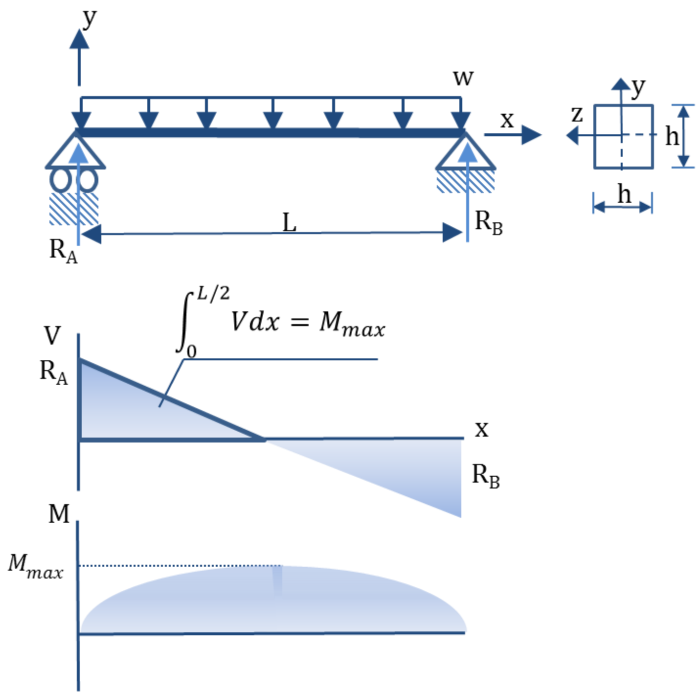


Figure 6-3

Solution:

Step 1 - Draw a free body diagram (shown in figure) and determine the reactions:

$\sum M = 0 \rightarrow R_A = R_B$ (due to symmetry)

$\sum F_y = 0 \rightarrow R_A = R_B = \frac{wL}{2}$

Step 2 – Calculate the functions for the internal shear force and bending moment curves in order to determine the maximum values of V(x) and M(x) that are to be used for design. The internal shear force function is determined by

$$dV = -wdx \rightarrow \int dV = \int -wdx \rightarrow V = -wx + C_1$$

The integration constant C_1 is calculated on basis of the boundary condition

$$V(x=0) = R_A \rightarrow -w \cdot 0 + C_1 = R_A \rightarrow C_1 = \frac{wL}{2}$$

In a similar manner, the internal bending moment function is determined by

$$dM = \int V dx \rightarrow M = \int -qx + \frac{wL}{2} dx = -\frac{q}{2}x^2 + \frac{wL}{2}x + C_2$$

The integration constant C_2 is calculated on basis of the boundary condition

$$M(x=0) = 0 \rightarrow -\frac{q}{2} \cdot 0^2 + \frac{wL}{2} \cdot 0 + C_2 = 0 \rightarrow C_2 = 0$$

In the last step, it has been utilized that the vertical supports are free of moment. The maximum values are

$V_{max} = \frac{wL}{2}$ for x=0 $\quad$ $M_{max} = \frac{1}{2}R_A\frac{L}{2} = \frac{1}{2}\frac{wL}{2}\frac{L}{2} = \frac{wL^2}{8}$ for x=L/2

Step 3: - Relate the stresses and internal shear force and bending moment using respectively the Grasshof's Formula and the Flexure Formula. Initially, design based on shear stress is considered:

$$\tau_{max,allow} = \frac{3}{2}\frac{V_{max}}{A} = \frac{3}{2}\frac{1}{h^2}\frac{wL}{2} \rightarrow h^2 > \frac{3}{2}\frac{1}{\tau_{max,allow}}\frac{wL}{2}$$

$$\rightarrow h > \sqrt{\frac{3}{2}\frac{1}{\tau_{max,allow}}\frac{wL}{2}} = \sqrt{\frac{3}{2}\frac{1}{2.5\frac{\mathrm{N}}{\mathrm{mm}^2}}\frac{1.5\frac{\mathrm{N}}{\mathrm{mm}}3000\mathrm{mm}}{2}} = 36.74 \text{ mm}$$

Finally, design based on normal (bending) stress is considered:

[Calculated example 6A continued …]

$$\sigma_{max,allow} = -\frac{M_z y}{I_z} = \frac{wL^2}{8}\frac{\left(-\frac{h}{2}\right)}{\frac{h^4}{12}} = \frac{wL^2}{\frac{16}{12}h^3} = \frac{3wL^2}{4h^3} \rightarrow h^3 > \frac{3wL^2}{4\sigma_{max,allow}}$$

$$\rightarrow h > \sqrt[3]{\frac{3wL^2}{4\sigma_{max,allow}}} = \sqrt[3]{\frac{3\cdot 1.5\frac{\text{N}}{\text{mm}}(3000\text{mm})^2}{4\cdot 15\frac{\text{N}}{\text{mm}^2}}} = 87.72 \text{ mm}$$

While the τ-based design can sustain the required load with h>36.74 mm, σ-based requires h>87.72 mm. The largest value of h obviously governs the design, so h>87.72 is required to ensure the structural integrity.

Calculated example 6B: Beam subjected to concentrated load

The shown I-beam has geometry given by the parameters L=4m, a=1,5m, h=400 mm, b=150 mm, t_w=10 mm and t_f=14 mm. The beam is subjected to a concentrated load P=75 kN. Determine the maximum values of normal stress due to bending and shear stress in the beam.

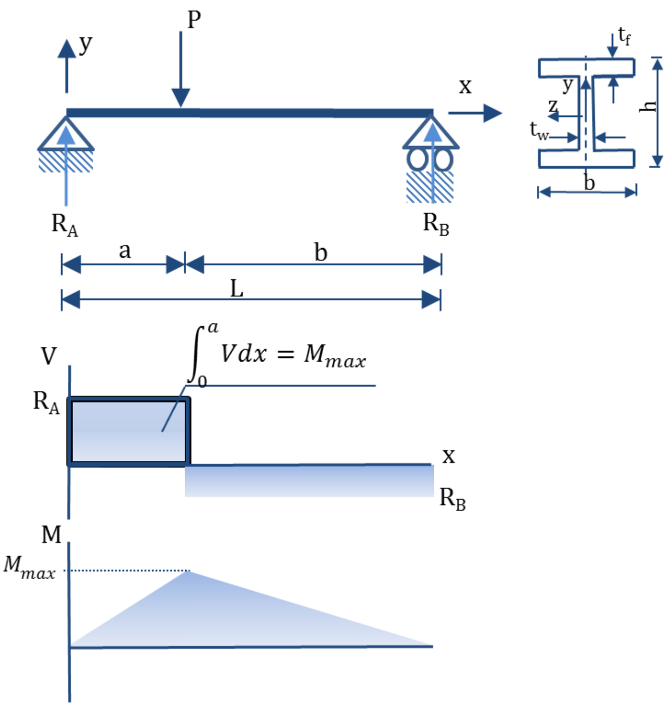


Figure 6-4

Step 1 – draw a free-body diagram and calculate the reactions:

$\sum M_z^A = 0 \rightarrow -aP + LR_B = O$

$\rightarrow R_B = \frac{a}{L}P$

$\sum F_y = 0 \rightarrow R_A + R_B - P = O$

$\rightarrow R_A = P - \frac{a}{L}P = \left(1 - \frac{a}{L}\right)P$

Step 2: The maximum values of M and V are given by

$V_{max} = \max\left(\frac{a}{L}P, \left(1-\frac{a}{L}\right)P\right) = \left(1-\frac{a}{L}\right)P$ for a<b

$= \left(1 - \frac{1500}{4000}\right) 75000 \text{ N} = 46875 \text{ N}$

$M_{max} = \int_0^a V dx = \int_0^a \left(1-\frac{a}{L}\right)P\, dx = \left[\left(1-\frac{a}{L}\right)Px\right]_0^a = \left(1-\frac{a}{L}\right)Pa$

$= \left(1 - \frac{1500}{4000}\right)(7500 \text{ N})(1500 \text{ mm}) = 70.312 \cdot 10^6 \text{Nmm}$

The maximum stresses are now calculated applying respectively the Flexure Formula and Grasshofs Formula. The moment of inertia I_z is by the methods presented in chapter 4 calculated to $I_z = 1.99\cdot 10^8$ mm^4.

$$\sigma_x = \frac{M_{max}\left(\frac{h}{2}\right)}{I_z} = \frac{70.312\cdot 10^6\text{Nmm}\cdot\left(\frac{400\text{mm}}{2}\right)}{1.99\cdot 10^8\text{mm}^4} = 70.52\,\frac{\text{N}}{\text{mm}^2}$$

$$\tau_{max} \approx \frac{V}{A_{web}} = \frac{46875 \text{ N}}{(400-2\cdot 14)\text{mm}\cdot 10 \text{ mm}} = 12.60\,\frac{\text{N}}{\text{mm}^2}$$

6.3. The principle of superposition

As long as the assumptions summarized in the introduction are not violated, the principle of superposition is valid. This means that the total internal forces causes by different actions can be decomposed as a sum of internal forces caused separately by different actions. We will do a calculated example to see how this works.

Calculated example 6C: Beam subjected to distributed and concentrated load

For the beam shown, calculate the maximum internal shear force and the maximum internal bending moment in the beam.

Solution:

Reviewing the beam loading, it is realized that the beam loadings can be decomposed into the two following cases: a) a simply supported beam subjected to a distributed load *w*, b) a simply supported beam subjected to a concentrated load at a distance a from the left support. V_{max} and M_{max} for these two cases were calculated in example 6B and 6C. V_{max} and M_{max} for the case shown can now be determined by superposion:

$$V_{max} = \frac{wL}{2} + \left(1 - \frac{a}{L}\right)P$$

$$M_{max} = \frac{wL^2}{8} + \left(1 - \frac{a}{L}\right)Pa$$

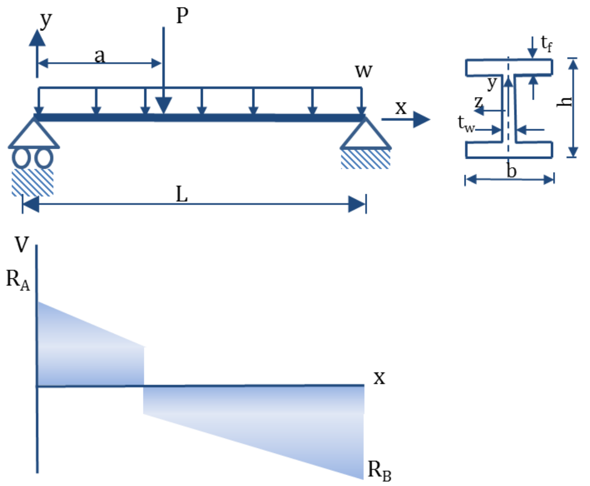
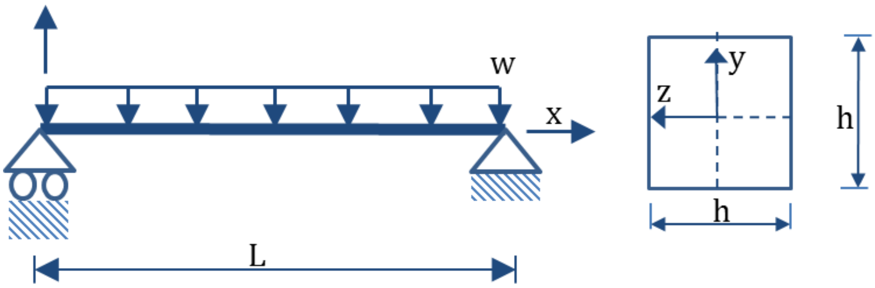


Figure 6-5

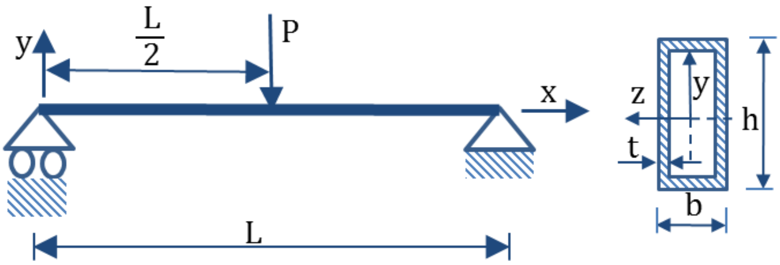
Problems

Problem 6.1



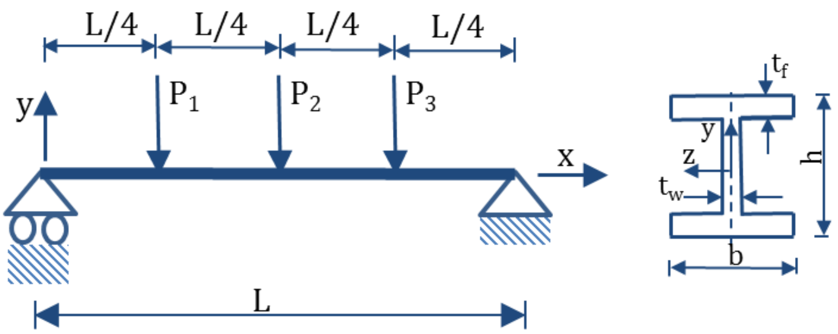
The beam shown is of length L=2.5 m. It is subjected to a distributed load w=100 kN/m. Calculate
a) the equations for the internal shear force and bending moment as functions of x. **b)** the required height of the beam if the allowable normal stress is specified to σ_{all}=100 N/mm², **c)** the required height of the beam if the allowable shear shear stress is specified to τ_{all}=35 N/mm²
Ans: **a) -, b)** h>167.34 mm, **c)** h>73,19 mm

Problem 6.2



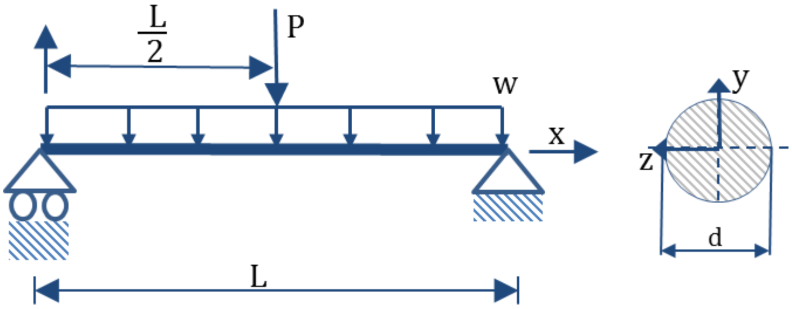
The geometry of the shown beam is given by h=300 mm, b=100 mm, t=10mm and L=2 m. The beam is loaded at the mid-point between the supports by a concentrated load P=120 kN. Determine by calculation **a)** the maximum internal bending moment in the beam, **b)** the maximum tensile normal stress due to bending in the beam, **c)** the maximum internal shear force in the beam, **d)** the maximum shear stress in the beam.
Ans: **a)** M_{max}=6000 Nm, **b)** σ_{max}=114.43, **c)** V_{max}=60 kN, **d)** τ_{max}=13.01 N/mm²

Problem 6.3



The geometry of the shown beam is given by h=400 mm, b=150 mm, t_w=10mm, t_f=14 mm and L=4 m. The beam is loaded as shown in the figure by three concentrated loads P=20 kN. Determine by calculation **a)** the maximum tensile normal stress due to bending in the beam, **b)** the maximum internal shear force in the beam, **c)** the maximum shear stress in the beam.
Ans: **a)** σ_{max}=40.12 N/mm², **b)** τ_{max}=8.06 N/mm²

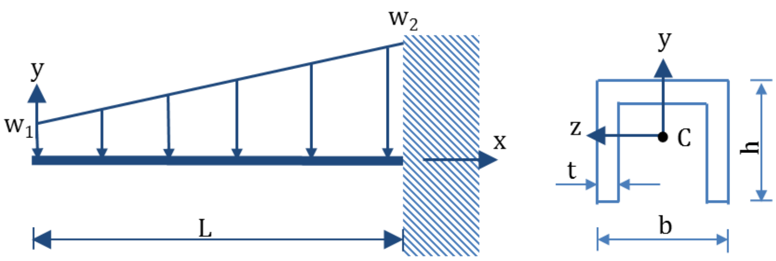
Problem 6.4



The beam shown is of length L=3.5 m and diameter d=200 mm and is subjected to a concentrated load P=10 kN and a distributed load w=22 kN/m. Determine by calculation **a)** the maximum tensile normal stress due to bending in the beam, **b)** the maximum shear stress in the beam.

Ans: **a)** σ_x=54.0 N/mm², **b)** τ=1.85 N/mm²

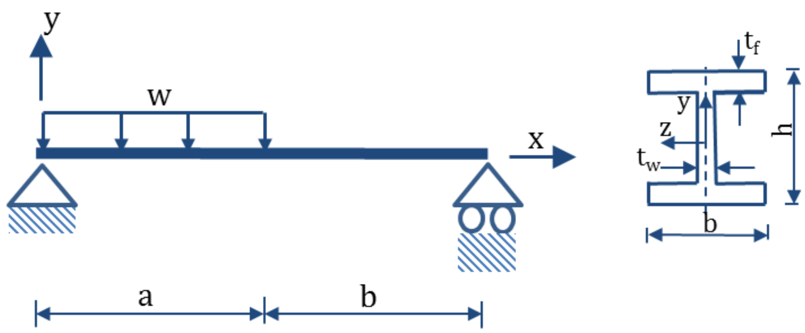
Problem 6.5



The cantilever beam shown is subjected to a distributed load increasing linearly from w_1=6 kN/m to w_2=18 kN/m. The beam is an U-profile (canal-profile) of length L=6 m with height h=280 mm, width b=220 mm and constant thickness t=20mm. Determine a) the maximum bending moment in the beam and it's position, b) the maximum bending stress in the beam and it's position.

Ans: a) M_{max}=-180·10⁶ Nmm, b) -258.9 N/mm²

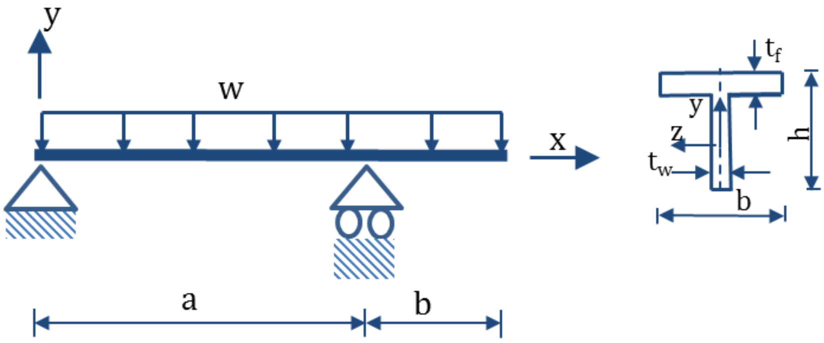
Problem 6.6



The beam shown is subjected to a distributed load w=16kN/m along the length a=1.5 m with b=1.0 m. The I-shaped cross-section is defined by the parameters h=140 mm, b=100 mm, t_w=10 mm and t_f=12 mm. Determine a) the maximum internal shear force and bending moment and their positions, b) The maximum shear stress and normal stress in the beam along with their positions

Ans: V_{max}= 16.80·10³N. M_{max}=8.820·10⁶ Nmm
σ_{max}=55.3 N/mm², τ_{max}=14.5 N/mm²

Problem 6.7



The beam shown is subjected to a distributed load w=3kN/m along the length a=2.4 m with b=1.2 m. The T-shaped cross-section is defined by the parameters h=180 mm, b=120 mm, t_w=14 mm and t_f=16 mm. Determine a) the maximum internal shear force and bending moment and their positions, b) The maximum shear stress and normal stress in the beam along with their positions

Ans: $V_{max}= -4.5\cdot10^3$N. $M_{max}=-2.160\cdot10^6$ Nmm

$\sigma_{max}=-19.45$ N/mm^2, $\tau_{max}=2.49$ N/mm^2