5. Shear stresses in beams

Contents

Nomenclature

M	Bending moment [Nmm]	h, r	Cross sectional dimensions [mm]
V	Shear force [N]	y, y_1	Distance, centroid to section [mm]
Q	Distributed load [N/mm]	d, d_1	Distance, global to local centroid [mm]
x	Longitudinal beam coordinate [mm]	b	Width [mm]
σ	Normal stress [Nmm^2]	A	Area
τ	Shear stress [Nmm^2]	Q	First order area moment [mm^3]
F	Force in general [N]	I	Second order area moment of inertia [mm^4]

5.1. Motivation

In this chapter, shear stresses in long and slender beams will be considered. From statics, we recall the two basic differential equations,

$$\frac{dM}{dx} = V, \quad \frac{dV}{dx} = -q$$

It follows from the first of the two equations that for a beam in pure bending (yielding a constant internal bending moment) the internal shear force V will be zero throughout the length of the beam. However, it is also clear that for a beam subjected to general loadings where the internal bending moment will vary throughout the beam length, the internal shear force will sustain a non-zero value. This shear force will produce a stress in the beam in addition to the normal stresses due to bending. If a cross section of a prismatic beam is considered, the shear force will be parallel to the section. The corresponding stress will therefore be a shear, see Figure 5-1.

For very long and slender members, the bending stresses will for most design cases be dominating, since thse usually are an order of magnitude larger than the shear stresses. However, the shorter the considered beam is, the more significant the shear stress component becomes. Since shear stresses are more critical for ductile materials than normal stresses, the total stress is underestimated if design is based only on calculation of normal stresses due to bending. This is our motivation for mastering the theory contained in the present chapter.

5.2. Derivation of Grasshof's formula

A rectangular beam in non-uniform bending will be considered, see Figure 5-1-A. Since the internal bending moment is not constant, it follows that the beam is subjected to an internal shear force V. A section of the beam above a line parallel to the z-axis and with a distance of y_1 from the centroid C in the y-direction will be considered. The section has centroid C_1 and area A_1. This section will now be considered along a small length of the beam *dx* so it forms a volume. The variation of stresses due to bending is shown in Figure 5-1-C. Since the upper face of the segment is free, it must be free of shear stress. However, since the stresses σ_1 and σ_2 are not equal, a stress component must act parallel to the lower face of the considered segment if equilibrium is to be maintained. Since this stress is parallel to the lower face, it is a shear component. Furthermore, we recall from the basic definition of shear, that the shear along the horizontal lower face will equal the shear along the vertical right face. Hence, if our objective is to obtain an expression for the stress acting in the cross-section, we may calculate the shear stress acting along the lower face, since the two shear terms will be equal.

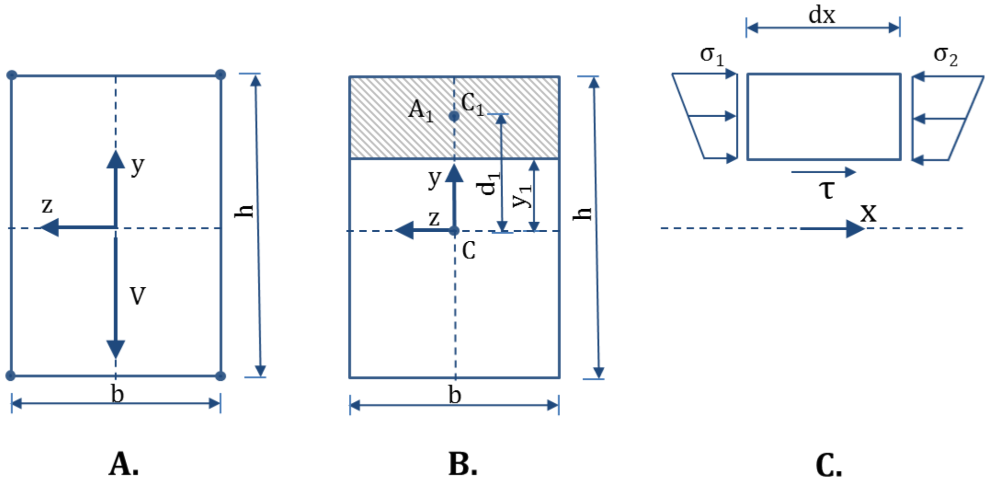

Figure 5-1 A rectangular beam in non-uniform bending

The shear force along the lower face of the segment can be calculated by $F_3 = F_2 - F_1$ in which F_1 and F_2 denote the forces corresponding to the respective normal stresses σ_1 and σ_2. From the relations between stresses and internal forces introduced in Chapter 1, we obtain the following expression for F_1

$$F_1 = \int_A \sigma_1 dA = \int_A \frac{My}{I_z} dA \tag{5-1}$$

By equivalent means, we obtain the following expression for F_2

$$F_2 = \int_A \sigma_2 dA = \int_A \frac{(M + dM)y}{I_z} dA \tag{5-2}$$

In the expressions derived above, the sign convention has been neglected, since we now manage those ourselves. The force along the lower face can now be obtained as

$$F_3 = F_2 - F_1 = \int_A \frac{dMy}{I_z} dA = \frac{dM}{I_z} \int_A y\, dA \tag{5-3}$$

We will now write this on the more compact form

$$F_3 = \frac{dM}{I_z} Q \quad \text{with } Q = \int_A y\, dA \tag{5-4}$$

In which Q denotes the first order area moment of the considered section. It is importance to note, that Q depends on at which height y_1 the section is considered, hence, where the shear stresses are required calculated. On the other hand, if the shear stress is assumed uniformly distributed in the z-direction of the cross section, the shear stress can alternatively be defined as

$$F_3 = \tau b dx \tag{5-5}$$

In which bdx represents the area of the lower section face where the shear stresses are calculated. Letting equation 5-4 equal equation 5-5, we obtain

$$\tau b dx = \frac{dM}{I_z} Q$$

$$\tau = \frac{dM}{dx} \frac{Q}{I_z b} = \frac{VQ}{I_z b} \tag{5-6}$$

This result is known as Grasshofs Formula and is generally valid for cross section with faces parallel to the y-axis.

The terms VQ/I_z is denoted the shear flow and is a measure for how must shear force that ‘passes’ through the considered segment. This is converted to a stress using the assumption that stresses are uniformly distributed in the z-direction. In order for this to hold, the dimensions of the considered beam must be small.

5.2.1 Shear stresses in a beam of solid rectangular cross-section

Grasshof’s formula can be simplified significantly for a solid rectangular cross-section. Applying the notation used in Figure 5-1, the area of the considered section is given by

$$A_1 = b\left(\frac{h}{2} - y_1\right) \tag{5-7}$$

Furthermore, the distance from the local to the global centroid is given by

$$d_1 = y_1 + \frac{1}{2}\left(\frac{h}{2} - y_1\right) = \frac{h}{4} + \frac{y_1}{2} \tag{5-8}$$

The first order area moment is now calculated. The following expression is obtained

$$Q = A_1 d_1 = b\left(\frac{h}{2} - y_1\right)\left(\frac{h}{4} + \frac{y_1}{2}\right)$$

$$= b\left(\frac{h^2}{8} - \frac{hy_1}{4} + \frac{hy_1}{4} - \frac{y_1^2}{2}\right) \tag{5-9}$$

$$= b\left(\frac{h^2}{8} - \frac{y_1^2}{2}\right)$$

Substituting this into Grasshof's formula, we obtain

$$\tau = \frac{VQ}{I_z b} = \frac{Vb\left(\frac{h^2}{8} - \frac{y_1^2}{2}\right)}{I_z b} \tag{5-10}$$

When reviewing this expression, we realize that the shear stress τ varies as a quadratic function of y_1 and that the maximum shear will occur in the horizontal plane containing the centroid for (in the middle of the beam cross section) for $y_1 = 0$. The stress distribution is visualized on Figure 5-2. The maximum stress are obtained to

$$\tau_{max} = \frac{V\left(\frac{h^2}{8}\right)}{\left(\frac{bh^3}{12}\right)} = \frac{3}{2}\frac{V}{hb} = \frac{3}{2}\frac{V}{A} \tag{5-11}$$

This expression is valid only for solid rectangular cross sections.

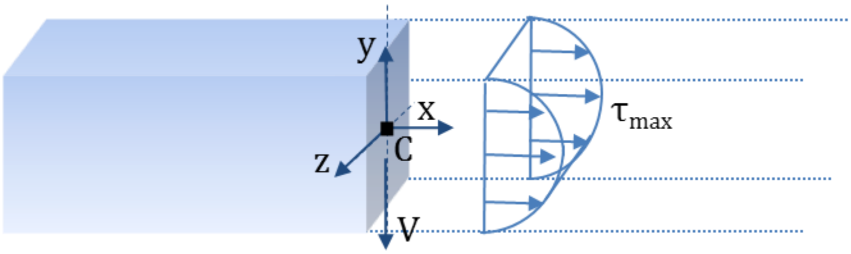

Figure 5-2 Distribution of shear stresses over a rectangular cross section

5.2.2 Shear stresses in a beam with I-shaped cross-section

The shear stress in an I-beam sustains a rather low value in the flanges and is larger in the web. The parabolic distribution of stresses over the web is in this case not dominating, and a fairly accurate expression appropriate for design purposes can be obtained by assuming the shear stresses uniformly distributed over the web.

$$\tau_{max} \approx \frac{V}{A_{web}} \tag{5-12}$$

You are to investigate the accuracy of this expression yourself in Problem 5.2.

5.1.2 Shear stresses in a solid circular beam

Circular cross sections are fairly problematic when applying the present framework for calculation of shear stress, since the sides of the cross-section are not parallel to the y-axis. However, the maximum shear stress will occur along the neutral axis where the faces actually locally are parallel to the y-axis. Hence, a general expression for the stress distribution cannot be obtained using Grasshof's Formula, but we may use it to assess the maximum shear stress for $y_1 = 0$. Applying an approach equivalent to what we did with the rectangular cross section, we obtain the following expression

$$\tau_{max} = \frac{4V}{3A} \tag{5-13}$$

We will do a calculated example to see how Grasshof's formula is applied.

Calculated example 5A: box shaped beam subjected to shear force

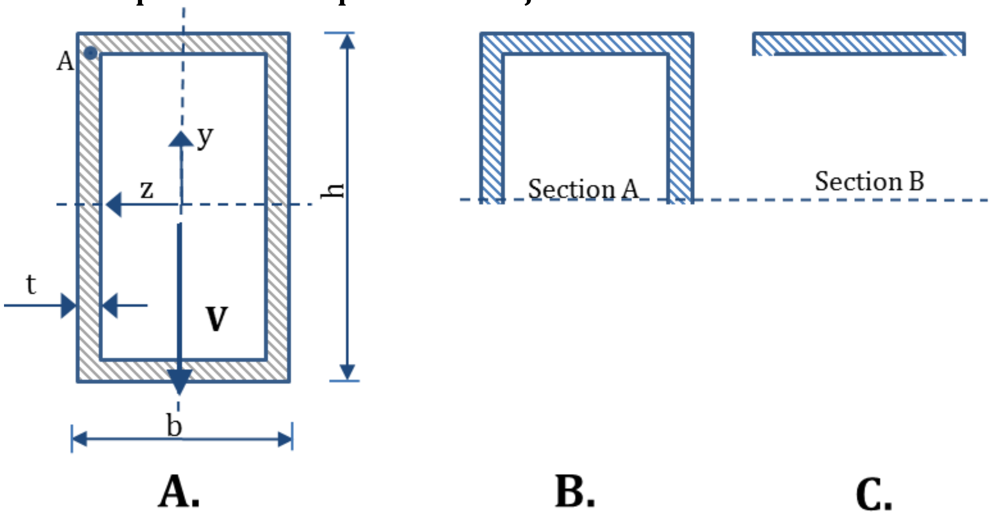

Figure 5-3

The geometry of the box shaped beam shown in Figure 5-3-A is given by the parameter set: h=190 mm, b=110 mm and t=13 mm. The beam is subjected to an internal shear force V=75 kN. Calculate **a)** the maximum shear stress in the beam, **b)** the shear stress in point A.

Solution

The stresses can be obtained using Grasshofs Formula (5-6). Initially, the moment of inertia is calculated by

$I_z = \frac{1}{12}(bh^3 - (b-2t)(h-2t)^3) = 31.99 \cdot 10^6 \mathrm{mm}^4$

a) The maximum shear stress will occur for the section having the largest Q/b ratio. This is for the

considered beam obtained for the cross-section along the line formed by y=0. The first order are moment of the section shown in Figure 5-3-B is given by

$Q = 2 \cdot \frac{h}{2} t \frac{h}{4} + (b-2t)t\left(\frac{h}{2} - \frac{t}{2}\right) = 0.214 \cdot 10^6 \mathrm{mm}^3$

Now applying Grasshofs Formula, we have

$$\tau_{max} = \frac{VQ}{I_z b} = \frac{75 \cdot 10^3 \mathrm{N} \cdot 0.214 \cdot 10^6 \mathrm{mm}^3}{31.99 \cdot 10^6 \mathrm{mm}^4 \cdot 2 \cdot 13\mathrm{mm}} = 19.29 \frac{\mathrm{N}}{\mathrm{mm}^2}$$

b) The first order area moment for the section shown in Figure 5-3-C is given by

$Q = bt\left(\frac{h}{2} - \frac{t}{2}\right) = 0.127 \cdot 10^6 \mathrm{mm}^3$

Hence, the shear stress is given by

$$\tau_A = \frac{VQ}{I_z b} = \frac{75 \cdot 10^3 \mathrm{N} \cdot 0.127 \cdot 10^6 \mathrm{mm}^3}{31.99 \cdot 10^6 \mathrm{mm}^4 \cdot 2 \cdot 13\mathrm{mm}} = 11.41 \frac{\mathrm{N}}{\mathrm{mm}^2}$$

Problems – Chapter 5

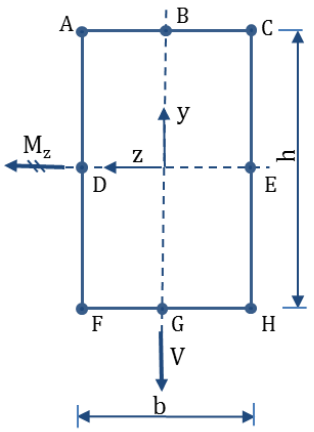

Figure P5.1

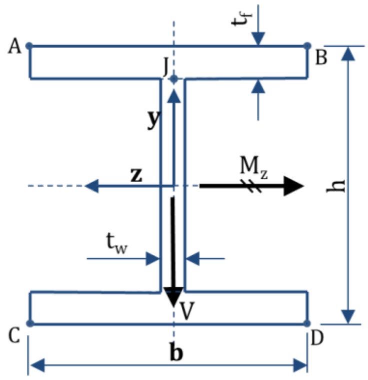

Figure P5.2

Problem 5.1
Figure P5.1 shows a rectangular beam with internal forces $M_z=250 \cdot 10^3$Nmm and $V=10 \cdot 10^3$N. The dimensions of the beam cross-section are given by h=30 mm and b=15 mm. **a)** calculate the maximum tensile stress in the beam, **b)** calculate the maximum shear stress in the beam and describe along which line this stress occurs (use the letters in figure 5.1)

Ans.: **a)** σ_x=111.1 N/mm², **b)** τ=33.3 N/mm²

Problem 5.2
The beam cross-section shown in Figure P5.2 is subjected to an internal bending moment with magnitude $|M_z|$= 75000 Nm and internal shear force with magnitude |V|=80 kN, both action in the directions shown on the figure. The geometry of the beam is given by h=325 mm, b=310 mm, t_w=15 mm and t_f=25 mm. **a)** calculate the maximum tensile normal stress due to bending and describe along which line this stress occurs, **b)** calculate the maximum shear stress in the beam cross-section using the approximate formula and describe along which line this stress occurs **c)** calculate the maximum shear stress in the beam cross-section using Grasshoffs Formula and compare the obtained result with the result obtained in question b), **d)** calculate the shear stress in the flange-weg junction (marked J) using Grasshoffs Formula.

Ans.: **a)** σ_x=32.5 N/mm², **b)** τ=19.4 N /mm², **c)** τ=18.5 N/mm², **d)** τ=16.5 N/mm²

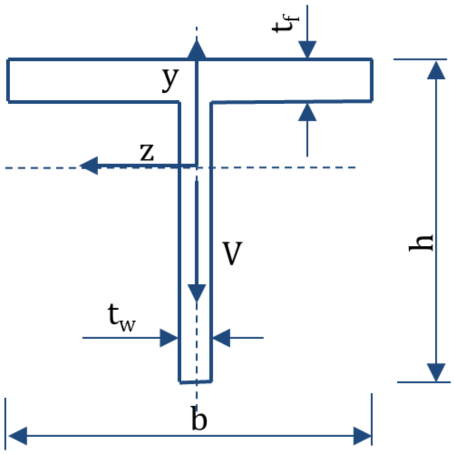

Figure P5.3

Problem 5.3
The beam shown in Figure P5.3 has geometry given by the following set of parameters: h=180 mm, b=120 mm, t_f=16 mm and t_w=14 mm. For V=15 kN, calculate **a)** the maximum shear stress in the beam cross-section, **b)** the shear stress in the flange-web junction.

Ans:
a) τ_{max}=8.31 N/mm², **b)** $\tau_{junction}$ =7.383 N/mm²