

4. Beams in pure bending

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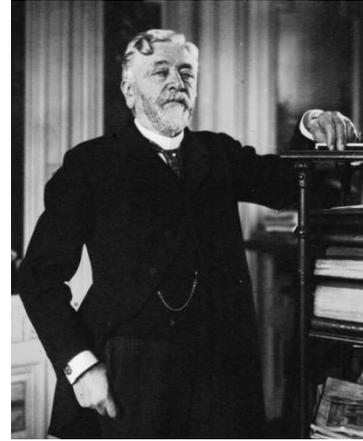
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Nomenclature

M	Bending moment [Nmm]	I	Second order area moment of inertia [mm ⁴]
L	Beam length, undeformed state [mm]	θ	Angle [rad]
L'	Beam length, deformed state [mm]	E	Module of elasticity [N/mm ²]
R	Radius of curvature [mm]	A	Cross sectional area [mm ²]
κ	Curvature [mm ⁻¹]	h	Height dimension [mm]
y,z	Distances to centroid in axis direction [mm]	B	Width dimension [mm]
r	Radius dimension [mm]	T	Thickness dimension [mm]

Awesome people in Engineering mechanics, ch. 4

Gustave Eiffel (1832 – 1923) was a French Engineer who after graduation from École Centrale des Arts et Manufactures in 1855 started his career as a bridge engineer by designing a high number of spectacular minimal designs. Later he founded his own companies and did not only design the famous Eiffel tower for the Paris World Exhibition in 1889, but also did the structural design of the Statue of Liberty and worked on the Panama Canal. During this project, he somehow managed to get tangled up in a scandal and was convicted for corruption, but despite this fact, we still remember him as one of the major pioneers in structural steel design



4.1. Introduction

In this chapter, beams subjected to end moments are considered. Our overall scope is to obtain a mathematical description for the stresses in beams in pure bending. Though derived for beams in pure bending as shown in Figure 4-1, we shall later see, that the derived expression, which is known as *the flexure formula*, actually with high accuracy can be applied for calculation of stresses in beams subjected to general loads.

The applied theory is known as *the Bernoulli-Euler-beam theory*, which is valid for long and slender beams. For short beams, an alternative but more complex mathematical description known as *the Timoshenko beam theory* is usually applied, but this theory is beyond the scope of the present chapter.

It is of great importance to note that the theory described in these notes is valid only for beams with cross-sections containing at least one axis of symmetry. If this is not the case, the expressions required to calculate stresses are complicated significantly. In order to obtain a mathematical description of the strain distribution throughout the cross-sections, it will in addition be assumed that cross-sections plane before loads are applied remain plane in the loaded state. This is in general valid for beams with a length to height ratio $L/h > 12...15$, which is a rule of thumb for when the theory provides accurate results. Hence, the analyses are limited to considerations of long and slender members. Furthermore, we assume that stresses are uniformly distributed through the width of the cross-sections. This assumption is very reasonable for tall members and less accurate for widths larger than height.

The material of the beam must be linear elastic, isotropic (meaning, that the material parameters are directionally independent) and homogeneous.

Our motivation for learning to analyze beams in bending should be obvious: virtually all systems in structural mechanics contain beams, which is an even more common structural element than trusses. The formulas derived in this chapter were according to engineering tribe lore first applied in large scale for designing structures like the Eiffel Tower, see Figure 4-2. Furthermore, bridges, power plants and cranes are other examples of structures which are dominated by beams as structural elements.

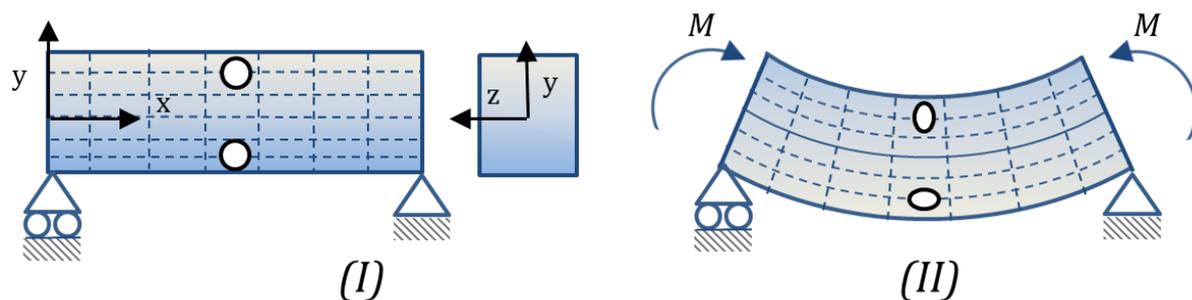


Figure 4-1, A: Beam with two holes subjected to a constant internal bending moment causing a stress state denoted pure bending, **B:** The corresponding mode of deformation, plane cross-sections will be assumed to remain plane.

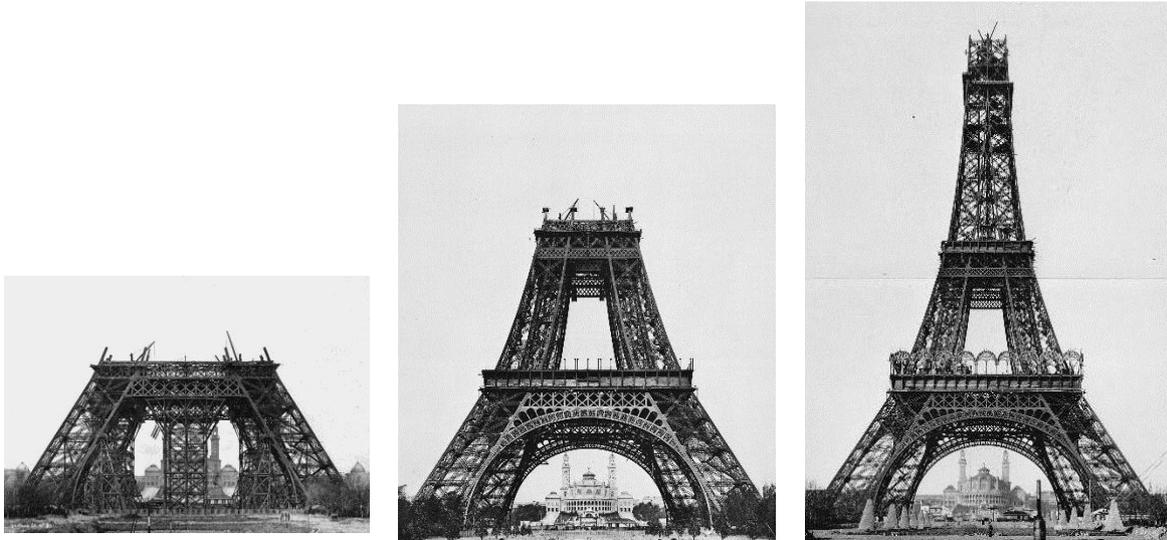


Figure 4-2 The construction of the Eiffel tower (Pictures from Wikimedia, public domain)

4.2. Static equilibrium of beam in pure bending

If a section of a prismatic beam subjected to an internal bending moment is considered, we recall from chapter 1, that the internal bending moment is related to the normal stress in the longitudinal x -direction by the following expression

$$M_z = - \int y \sigma_x dA \quad (4-1)$$

This does in a physical sense correspond to summing up longitudinal stress times distance to centroid plane for all infinitesimally small segments in the cross-section. However, since the stress distribution over the cross-section is still unknown, this expression cannot be applied for beam design without proper modification. The framework, which we will apply, is exactly the same as used when deriving equivalent expressions for axial loads and torsion: on basis of consideration of the deformations, the strain expressions can be derived and converted to stresses on basis of an appropriate constitutive relation, in the present case Hook's law.

Reviewing Figure 4-3, it is noted that we may visualize a positive bending moment as a vector with two short crossing lines. This corresponds to the rotational definition of a positive bending moment shown in Figure 4-1. Hence, both of the shown moments are positive, since they cause elongation of the lower beam face and compression of the upper beam face.

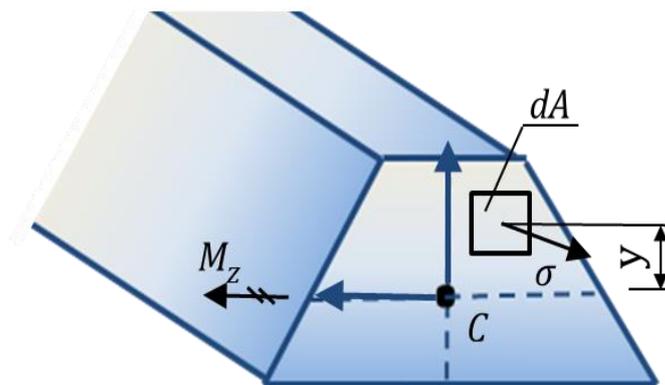


Figure 4-3 Cross-section of a beam subjected to a constant internal bending moment

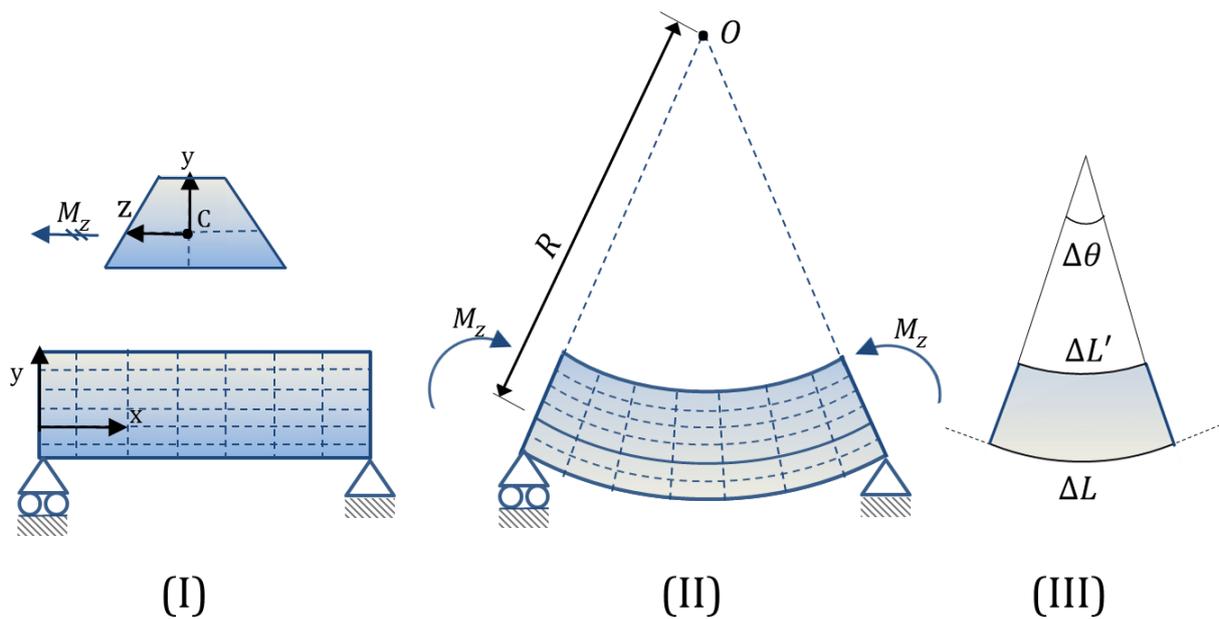


Figure 4-4 Prismatic beam in pure bending along with the geometrical parameters required for calculation of the longitudinal stress.

4.3. Mode of deformation and derivation of the longitudinal strain expression

A beam in pure bending generated by end moments will now be considered as shown in Figure 4-4. It is noticed that a positive moment (as defined in Figure 4-1) will compress the upper face of the beam and elongate the lower face. Hence, a positive internal moment produces a state of compressive stress and strain in the upper face and a state of tensile stress and strain in the lower face. If a cross-section of the beam is considered, the stresses and strains must therefore be zero at some point in the cross-section. The plane in the depth/width direction, where the length in the deformed state equals the length in the undeformed state is called *the neutral surface* or for a 2D cut *the neutral axis* and constitutes a set of points which for pure bending is free of stress and strain. The neutral axis will for beams with at least one axis of symmetry pass through the centroid of the cross-section. It is noted that for homogeneous materials, the centroid and center-of-gravity (COG) is located in the same point. For double symmetric cross-sections, the centroids are located at the intersection between the symmetry axes, see Figure 4-6. This is the case for cross-sections which are boxed shaped, circular and for I-shaped beams. For cross-sections with only one axis of symmetry (for example T-shaped beams), the centroid and thereby the neutral axis is located on this axis, but the unknown coordinate must be calculated by the weighted sum $\sum Ay / \sum A$.

The beam will be assumed deformed into a circular arc with radius R . This is for beams called the 'radius of curvature'. Furthermore, we will define an additional measure for how much the beam has been deformed, namely the curvature $\kappa = \frac{1}{R}$.

The length of the neutral axis L must be conserved in the deformed state, since this is free of strain. For a small arc length segment along the points m_1 and m_2 corresponding to an angle $d\theta$, we have $dL = Rd\theta$

However, if we move along a plane cross-section away from the centroid, lines originally parallel to the neutral axis will not be free of strain. Below the neutral axis these lines will be

elongated and above the neutral axis lines will be compressed, since the applied bending moment is positive. By arguments equivalent to the one above, the length dL' spanned by the points p and q is given by $dL' = R(d\theta - y)$. Recalling that the longitudinal normal strain is defined as deformation divided by original length, we obtain the following expression for the strain in the beam along a line parallel to the neutral axis, but off-set with a distance y .

$$\epsilon_x = \frac{dL' - dL}{dL} = \frac{Rd\theta - yd\theta - Rd\theta}{Rd\theta} = -\frac{y}{R} = -y\kappa \quad (4-2)$$

It is noted that the negative sign corresponds to the definition of where a positive moment causes elongation and compression. Furthermore, we note that all stresses due to pure bending are directed along the longitudinal x-axis. This allows us to apply Hook's law for uni-axial stress on basis of the strains obtained in Eq. 4-2. We then obtain the following expression

$$\sigma_x = E\epsilon_x = -Ey\kappa \quad (4-3)$$

All that is required now is to relate the stress in this expression to the internal bending moment.

4.4. Calculation of bending stresses: the flexure formula

Recalling the integral expression from chapter 1 which was introduced in Eq. 1, we may substitute the result obtained in the foregoing in Eq. 4-3 into this. We obtain the following

$$\begin{aligned} M_z &= - \int y\sigma_x dA = - \int y(-Ey\kappa) dA \\ &= \int E\kappa y^2 dA \end{aligned} \quad (4-4)$$

If assembling the geometric integral terms, which are specific for the considered cross-section, the following simplification can be conducted

$$M_z = EI\kappa \quad I_z = \int_A y^2 dA \quad (4-5)$$

In which the term I_z denotes the second order area moment of inertia, often just called the moment of inertia. Substituting the obtained result in Eq 4-5 into Eq. 4-3, we obtain

$$\sigma_x = -Ey\kappa = -Ey \frac{M_z}{EI} = -\frac{M_z y}{I_z} \quad (4-6)$$

The negative sign in this equation shows that stresses are positive at the lower face of the beam (which is elongated when subjected to a positive internal moment) and negative at the upper face of the beam (compressed when subjected to a positive bending moment). This result is known as 'the flexure formula' and the stresses as function of height above the neutral axis, can be observed linearly distributed, see Figure 4-5.

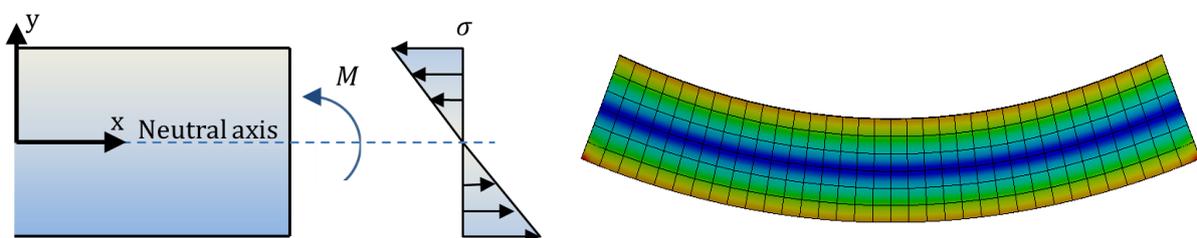


Figure 4-5 Linear distribution of bending stresses over the cross-section of a beam subjected to pure bending (figure to the right only shows stress magnitudes, while figure to the left shows the sign of the stress)

4.4.1 Calculation of the centroid position

For a cross section of homogeneous materials the center of gravity and centroid will coincide. Furthermore, it can be shown that the neutral axis will pass through the centroid of the considered cross section. We recall from mathematics and physics, that the centroid of a cross section is defined by the equations

$$\bar{x} = \frac{\sum A_i \bar{x}_i}{\sum A_i} \quad \bar{y} = \frac{\sum A_i \bar{y}_i}{\sum A_i} \quad (4-7)$$

Examples of centroids for various cross sections are shown in Figure 4-6. It is noted, that if cross sections contain an axis of symmetry, the centroid will be located on this axis. Hence, if a cross section is double symmetrical, the centroid is located in the middle of the cross section.

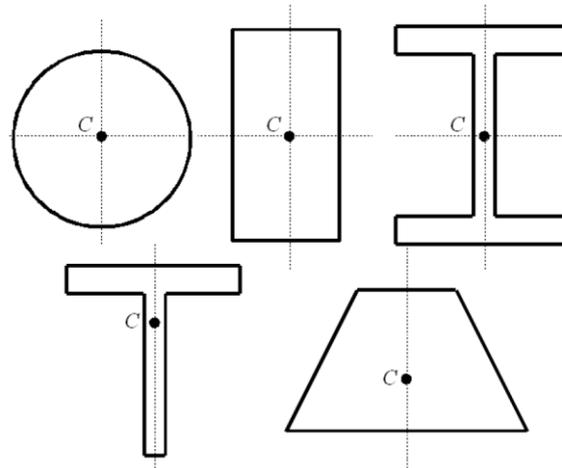


Figure 4-6 Centroid positions for different cross sections

Example 4A: centroid of a beam with T-shaped cross-section

Calculate the position of the centroid for the T-shaped beam shown in Figure 4-7.

Solution

Since the beam is symmetric around the y-axis, the centroid is horizontally located in the middle of the beam. Hence

$$\bar{z} = \frac{b}{2}$$

The vertical position of the centroid is calculated by

$$\begin{aligned} \bar{y} &= \frac{\sum A_i \bar{y}_i}{\sum A_i} = \frac{A_1 \bar{y}_1 + A_2 \bar{y}_2}{A_1 + A_2} \\ &= \frac{(t_w(h-t_f))\frac{h-t_f}{2} + (t_f b)\left(h - \frac{t_f}{2}\right)}{(t_w(h-t_f)) + (t_f b)} \end{aligned}$$

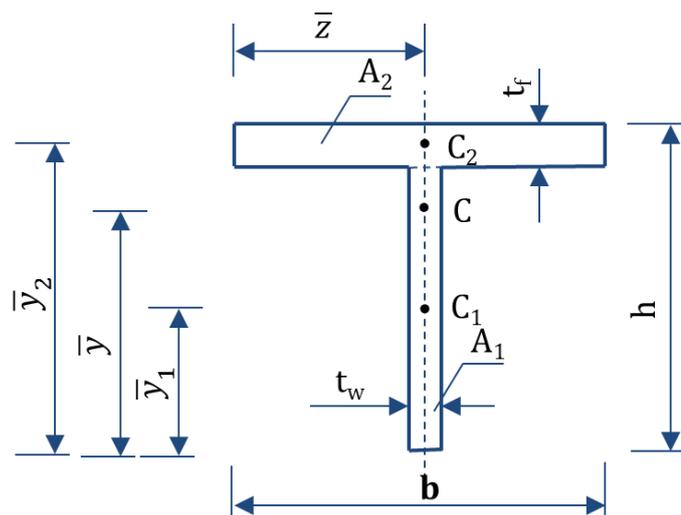


Figure 4-7

4.4.2 Calculation of the second order area moment inertia

The second order area moment of inertia is a specific geometric property of a cross section. Opposite the polar moment of inertia, we may base calculation of bending stresses on moments of inertia obtained for a wide range of different geometries. A large number of standard formulas for moments of inertia are available and can be applied for stress calculation in mechanical design. In this section, it will be considered how these expressions were derived. The double integrals are evaluated in exactly the same fashion as when we calculated the polar moment of inertia in Chapter 3. We will first consider one integration variable constant and perform the integration with respect to the second variable, then consider the second variable constant and perform the integration with respect to the first variable. For a rectangular cross section with height h and width b we obtain

$$\begin{aligned}
 I_{z,rect} &= \int_A y^2 dA \\
 &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} y^2 dz dy &= \int_{-\frac{h}{2}}^{\frac{h}{2}} [y^2 z]_{-\frac{b}{2}}^{\frac{b}{2}} dy \\
 &= \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 b dy = \left[\frac{y^3 b}{3} \right]_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{b}{3} \left(\frac{h^3}{8} - \left(-\frac{h^3}{8} \right) \right) \\
 &= \frac{1}{3} \frac{1}{4} b h^3 \\
 &\rightarrow I_{z,rect} = \frac{b h^3}{12}
 \end{aligned} \tag{4-8}$$

When calculating the second order area moment of inertia for a circle, it is convenient to apply polar coordinates. We write

$$x = \rho \cos\theta \quad y = \rho \sin\theta \tag{4-9}$$

The integral will then be on the form

$$I_{z,circ} = \int_0^{2\pi} \int_0^r (\rho \sin\theta)^2 \rho d\rho d\theta \tag{4-10}$$

This time we will spare us selves for the calculations and just state, that the result of the integration is

$$I_{z,circ} = \frac{\pi}{4} r^4 \tag{4-11}$$

It is of great importance to note that area moments of inertia are additive when calculated around the same axis, see Figure 4-9, A. This is practical when calculating the moment of inertia for complex geometries, since these may be decomposed into a sum of contributions from different less complex segments. Finally, we will consider calculation of the area moment around an axis $I_{z'}$ that does not pass through the centroid as I_z does, but is parallel to I_z . This may be conducted by the so-called parallel axis theorem given by

$$I_{z'} = I_z + A \cdot d^2 \tag{4-12}$$

In this expression, A is the cross sectional area and d is the distance between the two axes. How this exactly works will be demonstrated in the following example.

4.4.3 Moment of inertia of an I-beam

The procedure for calculation of moment of inertia for an I-beam is shown in Figure 4-9. Initially, we apply the result from equation 4-8 to obtain the moment of inertia for the web segment which is rectangular shaped

$$I_{z,web} = \frac{1}{12} t_w (h - 2t_f)^3 \quad (4-13)$$

We may now in the same manner calculate the moment of inertia for a single flange. However, this would give us a value for the moment of inertia for an axis passing through the centroid of the flange. Since the cross section centroid is located in the middle of the cross-section, the parallel axis theorem is applied in order to obtain the moment of inertia calculated around the neutral axis

$$I_{z,flange} = \frac{1}{12} b t_f^3 + (b \cdot t_f) \left(\frac{h}{2} - \frac{t_f}{2} \right)^2 \quad (4-14)$$

The total moment of inertia is now given by the moment of inertia for the web plus two times the moment of inertia of the flange calculated around the neutral axis

$$I_z = \frac{1}{12} t_w (h - 2t_f)^3 + 2 \left(\frac{1}{12} b t_f^3 + (b \cdot t_f) \left(\frac{h}{2} - \frac{t_f}{2} \right)^2 \right) \quad (4-15)$$

Calculated example 4B: simple bending of a beam with I-shaped cross-section

The beam shown in Figure 4-8 has geometry given by the parameter set $b=100$ mm, $h=200$ mm, $t_w=5$ mm and $t_f=6$ mm. The beam is subjected to an internal bending moment $M_z=10.0 \cdot 10^3$ Nm. Calculate **a)** the maximum tensile normal stress due to bending, **b)** the maximum compressive normal stress due to bending. Describe where in the cross-section this stress will occur.

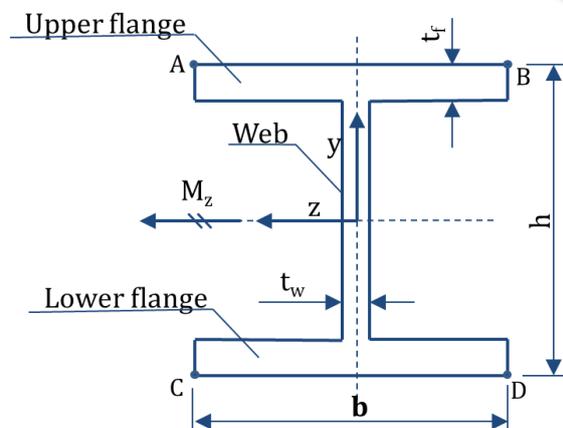


Figure 4-8

Solution

In order to apply the flexure formula, the second

order area moment of inertia of the beam is required. This is in accordance with equation (4-15) given by

$$\begin{aligned} I_z &= \frac{1}{12} t_w (h - 2t_f)^3 + 2 \left(\frac{1}{12} b t_f^3 + (b \cdot t_f) \left(\frac{h}{2} - \frac{t_f}{2} \right)^2 \right) \\ &= \frac{(5\text{mm})(200\text{mm} - 2 \cdot 7\text{mm})^3}{12} \\ &\quad + 2 \left(\frac{(100\text{mm})(7\text{mm})^3}{12} + (100\text{mm})(7\text{mm}) \left(\frac{200\text{mm}}{2} - \frac{7\text{mm}}{2} \right)^2 \right) = 1.572 \cdot 10^7 \text{mm}^4 \end{aligned}$$

The maximum tensile stress will, since $M_z > 0$, occur for $y = -h/2$ along the lower face of the beam constituted by line CD. Applying the flexure formula, the following result is obtained

$$\sigma_x^{CD} = - \frac{M_z \left(-\frac{h}{2} \right)}{I_z} = \frac{10 \cdot 10^6 \text{Nm} \frac{200\text{mm}}{2}}{1.572 \cdot 10^7 \text{mm}^4} = 63.6 \frac{\text{N}}{\text{mm}^2}$$

The maximum compressive stress is since since $M_z > 0$ encountered at the upper face of the beam constituted by line AB for $y = h/2$. This stress can be calculated by

$$\sigma_x^{AB} = - \frac{M_z \left(\frac{h}{2} \right)}{I_z} = - \frac{10 \cdot 10^6 \text{Nm} \frac{200\text{mm}}{2}}{1.572 \cdot 10^7 \text{mm}^4} = -63.6 \frac{\text{N}}{\text{mm}^2}$$

In example 4B, It can be observed that $\sigma_x^{CD} = -\sigma_x^{AB}$ since the beam is double symmetric, so the distance from the centroid to the position of maximum tensile stress equals the distance from the centroid to the position of the maximum compressive stress. For beams which are not double symmetric (like the T-shaped beam in example 4A), this would obviously not hold.

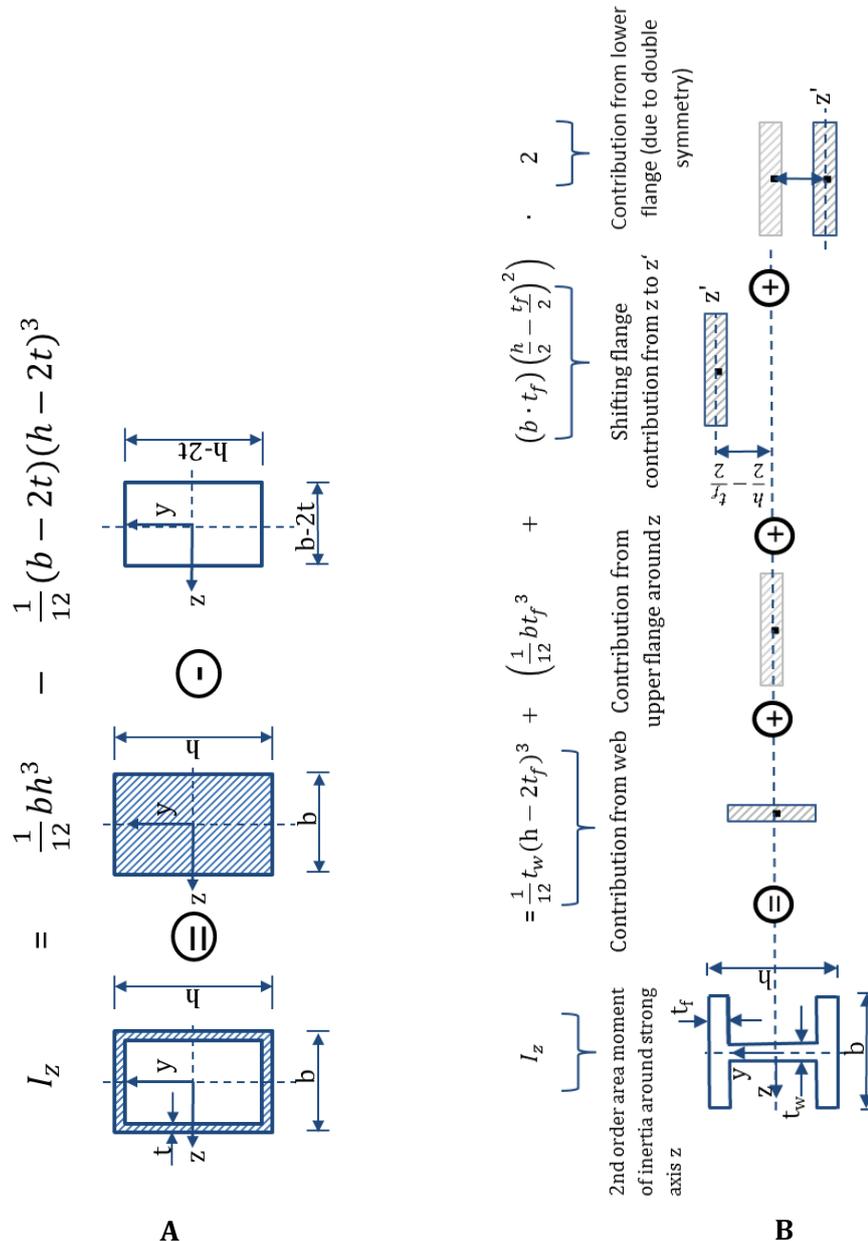


Figure 4-9 Illustration of the procedure for calculation of moments of inertia

4.5 Multi-axial bending

If a prismatic beam is subjected to bending along two coordinate axes along with axial loads, the stresses can be obtained by superposition. Under these circumstances, the flexure formula can be expanded to the more general form

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} + \frac{F_x}{A} \quad (4-16)$$

The superposition principle and separate stress distributions are shown in Figure 4-10

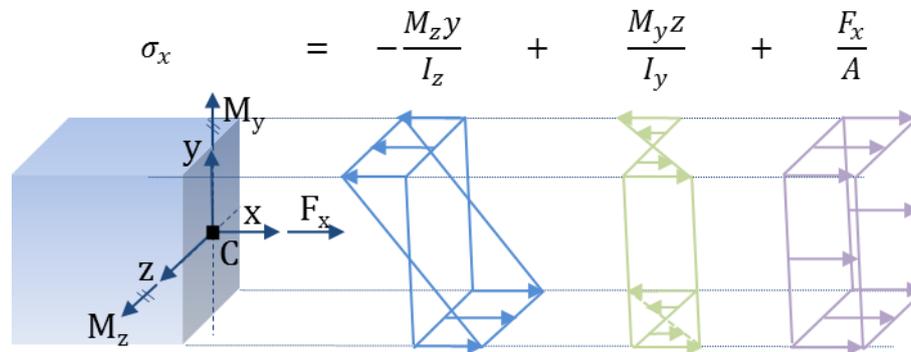


Figure 4-10 Multi-axial bending and axial loading of prismatic beam

Example 4C demonstrates how analysis of beams in multi-axial bending is performed.

4.6 Oblique bending – bending of non-prismatic beams

The term *oblique bending* can be defined as a state of deformation in a beam, by which the applied loads act in a plane different from the plane in which the deformations occur. The theoretical framework required to obtain a mathematical description of this phenomenon is constituted by the Bernoulli-Euler beam theory. However, the theory is formulated on a more general form than required for modeling of in-plane bending (also known as pure bending), which can be considered a special case within the general theoretical framework.

The physics of oblique bending are easily demonstrated by experimental means. If bending is added to a beam with axi-symmetric cross-section along an axis not coinciding with an axis of symmetry as shown in Figure 4-12, the deformations will occur along both axes in the Cartesian-coordinate system. However, it is easily recognized, that it is beneficial to project the moments onto the axes of symmetry and apply these as basis for calculation of the deformations. This may not be the case for non-symmetric cross-sections, for example the L-shaped profile shown in Figure 1-B.

It will now for a symmetric and a non-symmetric cross-section be explained how a rotation of the coordinate-system from the Cartesian xyz -system to what is usually referred to as principal coordinates (denoted $\eta\zeta$) enables de-coupling of the bending axes, so bending stresses and deformations may be calculated by superposition of two separate cases of simple bending.

Calculated example 4C: Hollow rectangular beam in multi-axial bending

The hollow rectangular beam shown in Figure 4-11 has geometry given by the parameter set $h=175$ mm, $b=100$ mm and $t=10$ mm. The beam is subjected to the internal bending moment $M=15 \cdot 10^3$ Nm which is oriented with an angle $\theta=65$ deg with respect to the z -axis. Calculate the stresses in the four corner points (A,C,F and H).

Solution

The stresses can be calculated using the generalized flexure formula in equation (4-16). The calculation requires the moments of inertia around both the y - and z -axis. These are:

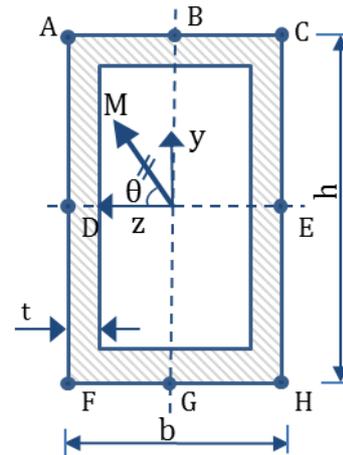


Figure 4-11

$$I_z = \frac{1}{12}bh^3 - \frac{1}{12}(b-2t)(h-2t)^3$$

$$= \frac{(100\text{mm})(175\text{mm})^3}{12} - \frac{(100\text{mm}-2 \cdot 10\text{mm})(175\text{mm}-2 \cdot 10\text{mm})^3}{12} = 1.984 \cdot 10^7 \text{ mm}^4$$

$$I_y = \frac{1}{12}hb^3 - \frac{1}{12}(h-2t)(b-2t)^3$$

$$= \frac{(175\text{mm})(100\text{mm})^3}{12} - \frac{(175\text{mm}-2 \cdot 10\text{mm})(100\text{mm}-2 \cdot 10\text{mm})^3}{12} = 7.970 \cdot 10^6 \text{ mm}^4$$

The stresses in the four corner points can now be obtained:

$$\sigma_x^A = -\frac{M \cos \theta \left(\frac{h}{2}\right)}{I_z} + \frac{M \sin \theta \left(\frac{b}{2}\right)}{I_y}$$

$$= -\frac{15 \cdot 10^6 \text{ Nmm} \cdot \cos 65^\circ \left(\frac{175\text{mm}}{2}\right)}{1.984 \cdot 10^7 \text{ mm}^4} + \frac{15 \cdot 10^6 \text{ Nmm} \cdot \sin 65^\circ \left(\frac{100\text{mm}}{2}\right)}{7.970 \cdot 10^6 \text{ mm}^4} = 57.3 \frac{\text{N}}{\text{mm}^2}$$

$$\sigma_x^C = -\frac{15 \cdot 10^6 \text{ Nmm} \cdot \cos 65^\circ \left(\frac{175\text{mm}}{2}\right)}{1.984 \cdot 10^7 \text{ mm}^4} + \frac{15 \cdot 10^6 \text{ Nmm} \cdot \sin 65^\circ \left(-\frac{100\text{mm}}{2}\right)}{7.970 \cdot 10^6 \text{ mm}^4} = -113.3 \frac{\text{N}}{\text{mm}^2}$$

$$\sigma_x^F = -\frac{15 \cdot 10^6 \text{ Nmm} \cdot \cos 65^\circ \left(-\frac{175\text{mm}}{2}\right)}{1.984 \cdot 10^7 \text{ mm}^4} + \frac{15 \cdot 10^6 \text{ Nmm} \cdot \sin 65^\circ \left(\frac{100\text{mm}}{2}\right)}{7.970 \cdot 10^6 \text{ mm}^4} = 113.3 \frac{\text{N}}{\text{mm}^2}$$

$$\sigma_x^H = -\frac{15 \cdot 10^6 \text{ Nmm} \cdot \cos 65^\circ \left(-\frac{175\text{mm}}{2}\right)}{1.984 \cdot 10^7 \text{ mm}^4} + \frac{15 \cdot 10^6 \text{ Nmm} \cdot \sin 65^\circ \left(-\frac{100\text{mm}}{2}\right)}{7.970 \cdot 10^6 \text{ mm}^4} = -57.3 \frac{\text{N}}{\text{mm}^2}$$

The normal stress in a beam σ_x of arbitrary cross-section with area A due to bending moments M_y and M_z and normal force N_x is given by

$$\sigma_x = \frac{M_y I_{yz} - M_z I_{yy}}{(I_{yy} I_{zz} - I_{yz}^2)} y + \frac{-M_y I_{zz} + M_z I_{yz}}{(I_{yy} I_{zz} - I_{yz}^2)} z + \frac{N_x}{A} \quad (4-17)$$

Equation 4-17 is derived by fixing the origin in of the xyz -system in the centroid of the cross-section and assuming strains uniformly distributed. The moments of inertia required for calculation of the normal stress are given by

$$I_{yy} = \int_A z^2 dA \quad I_{zz} = \int_A y^2 dA \quad I_{yz} = I_{zy} = - \int_A yz dA \quad (4-18)$$

In Equation 4-18, the two first terms constitute the second order areas moments of inertia applied for calculations of stresses and deformations due to in-plane bending. However, the last equation I_{yz} is a mixed term known as the product moment of inertia. This term prohibits us

from splitting the calculation of the normal stress in two de-coupled problems. Furthermore, it complicates calculation of the deformations significantly. Since the mixed term has quite unfortunate consequences for the theoretical framework, it is desirable to eliminate it. This can be carried out by a coordinate transformation simply by rotating the coordinate system around the x-axis until the product term has vanished. The procedure is entirely equivalent to the transformation utilized for construction of Mohr's circle, which by rotation transforms a stress state to principal coordinates where shear terms vanish. Having the xyz -system prescribed, the angle of rotation ϕ required to obtain the principal axes of inertia is given by

$$\tan(2\phi) = \frac{2I_{yz}}{I_{yy} - I_{zz}} \quad (4-19)$$

It follows from equation 4-18, that the product moment of inertia vanishes when calculated around principal axes of inertia, which due to the squared terms is not the case for the remaining second order moments. Hence, axes of symmetry in a cross-section are principal axes.

Since the system of principal axes are constructed by rotation in the cross-sectional yz -plane, the principal coordinate system also has origin in the centroid. The neutral axes can in absence of a normal force be determined by equation (4-17) as the axis where the stresses are zero. This provides the following slope

$$\tan\theta = \frac{M_y I_{yz} - M_z I_{yy}}{M_y I_{zz} - M_z I_{yz}} \quad (4-20)$$

While deformations will occur in a plane perpendicular to the neutral axes, the maximum stresses will as usual occur in the fibers with the furthest distance from the neutral axes. The principal moments of inertia are determined by the conventional 2-D transformation formulas for 2nd order tensors. After transformation to principal coordinates, normal stresses and deformation can be calculated by superposition of two decoupled problems and equation 1 reduces to

$$\sigma = -\frac{M_z y}{I_{zz}} + \frac{M_y z}{I_{yy}} + \frac{N_x}{A} \quad (4-21)$$

This is the flexure formula on the form we have applied this far.

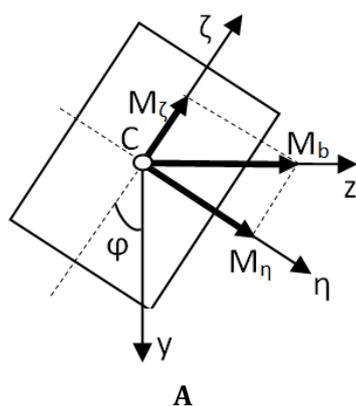


Figure 4-12 Oblique bending of rectangular cross-section

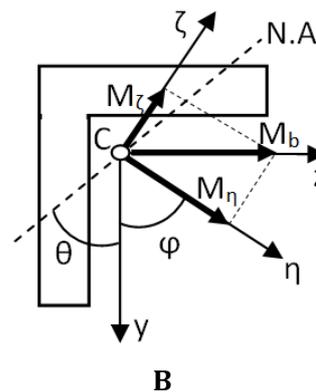


Figure 4-13 Oblique bending of L-shaped cross-section

Problems

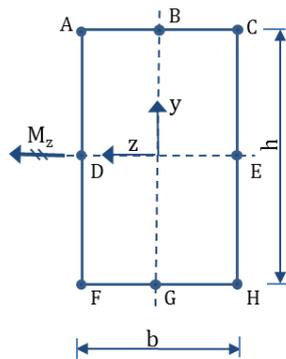


Figure P4.1

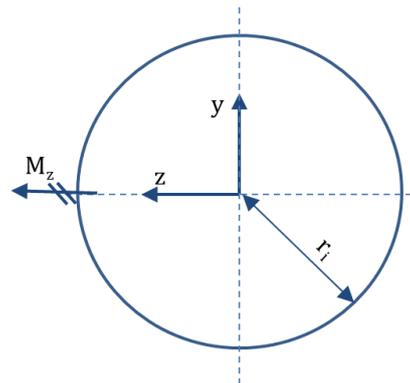


Figure P4.2

Problem 4.1

Figure P4.1 shows a rectangular wooden beam in pure bending due to the internal bending moment $M_z = 1.0 \cdot 10^6 \text{ Nmm}$. The dimensions of the beam cross-section are given by $h = 150 \text{ mm}$ and $b = 100 \text{ mm}$. **a)** Calculate the largest tensile stress in the beam and describe along which line this stress occurs (use the letters in figure 4.1), **b)** Calculate the largest compressive stress and describe along which line this would occur.

The maximum allowable normal stress in the beam is given by $\sigma_{\text{allow}} = 7.5 \text{ N/mm}^2$. **c)** Determine the largest internal bending moment the beam can sustain.

Ans: a) tension: $\sigma = 2.67 \text{ N/mm}^2$, b) compression: $\sigma = -2.67 \text{ N/mm}^2$, c) $2.813 \cdot 10^6 \text{ Nmm}$

Problem 4.2

The circular beam shown in figure P4.2 is subjected to a pure bending moment $M_z = 1500 \text{ Nm}$. The maximum allowable stress in the beam is given by $\sigma_y = 355 \text{ N/mm}^2$. Determine the required radius of the beam.

Ans: a) $r = 17.5 \text{ mm}$

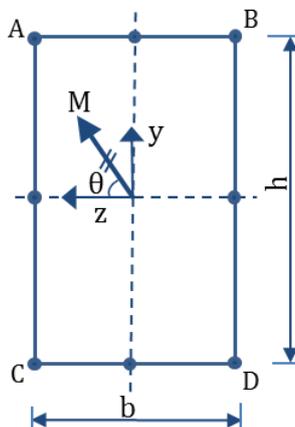


Figure P4.3

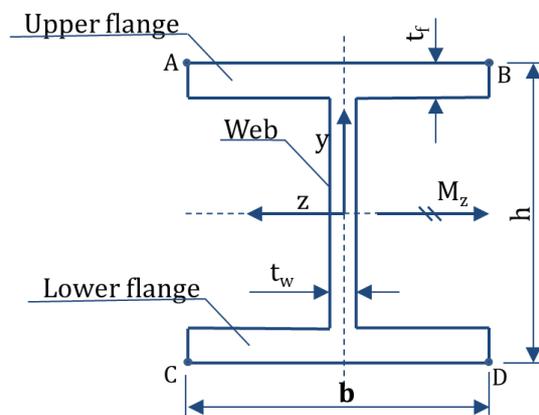


Figure P4.4

Problem 4.3

The beam shown in figure P4.3 is subjected to a moment $M = 20 \text{ kNm}$ oriented with an angle $\theta = 35 \text{ deg}$. with respect to the z-axis. The beam is of height $h = 200 \text{ mm}$ and width $b = 100 \text{ mm}$. **a)** Calculate the stresses in the four corner points (A, B, C and D) of the rectangular beam, **b)** could the applied method for stress calculation be directly applied to an I-shaped beam?, **c)** could the applied method be directly applied for an L-shaped beam?

Ans: a) $\sigma_A = 9.8 \text{ N/mm}^2$, $\sigma_B = -59.0 \text{ N/mm}^2$, $\sigma_C = 59.0 \text{ N/mm}^2$, $\sigma_D = -9.8 \text{ N/mm}^2$

Problem 4.4

Figure P4.4 shows an I-shaped beam with $t_f=7$ mm, $t_w=5$ mm, $h=200$ mm and $b=100$ mm. The internal bending moment is of magnitude $|M_z|=8 \cdot 10^6$ Nmm acting as shown in the figure. **a)** Calculate the maximum tensile bending stress in the beam. Where in the beam cross-section does this stress occur?

The beam is made of structural steel with yield stress $\sigma_Y=235$ N/mm². **b)** What is the maximum internal bending moment the beam can sustain without plasticity occurring anywhere the cross-section?

Ans: a) $\sigma=50.9$ N/mm², b) $M<36.9 \cdot 10^6$ Nmm

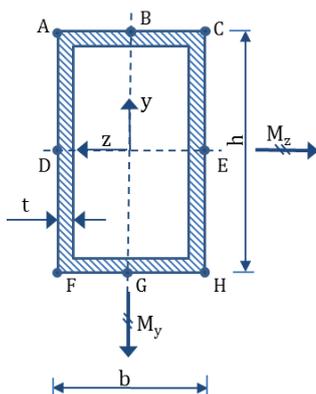


Figure P4.5

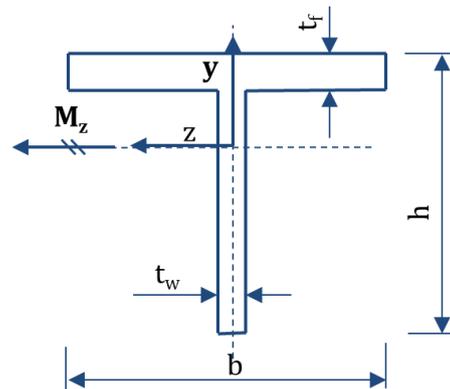


Figure P4.6

Problem 4.5

The beam cross-section shown on Figure P4.5 is subjected to internal bending moments with magnitudes $|M_z|= 13.5$ kNm and $|M_y|=23.38$ kNm and directions specified in the figure. The cross-section is of height $h=225$ mm, width $b=125$ mm and uniform thickness $t=10$ mm. Calculate the stresses in the four corner points (A,C, F and H) of the rectangular beam.

Ans: $\sigma_A=-51.7$ N/mm², $\sigma_C=121.9$ N/mm², $\sigma_F=-121.9$ N/mm², $\sigma_H=51.7$ N/mm²

Problem 4.6

The beam shown in Figure P4.6 has geometry given by the following set of parameters: $h=180$ mm, $b=120$ mm, $t_f=16$ mm and $t_w=14$ mm. For $M_z=3.5 \cdot 10^3$ Nm, calculate **a)** the normal stress due to bending at the top face of the beam, **b)** the normal stress due to bending at the bottom face of the beam.

Ans: a) $\sigma_{top}=-14.61$ N/mm², b) $\sigma_{bottom}=31,52$ N/mm²
($\bar{y}=122.99$ mm from bottom, $I_z=13.66 \cdot 10^6$ mm⁴)