3. Shafts in torsion

Contents

Nomenclature

T	Torsional moment/torque [Nmm]	G	Shear modulus [N/mm^2]
r	Shaft radius [mm]	J	Polar moment of inertia [mm^4]
ρ	Internal shaft radius, $r>\rho$ [mm]	τ	Shear stress [N/mm^2]
γ	Shear strain [rad]	θ	Angles in general [rad]
φ	Angle of twist [rad]	L	Length [mm]

3.1 Introduction

Torsion refers to a state of stress induced by twisting of a slender member caused by moments around the longitudinal axis. This direction will in the following again be denoted the *axial direction* and furthermore the torsional moments are usually referred to as *torques*. The simplest case of torsion imaginable is constituted by a shaft with constant diameter subjected to a torque in each end. This is shown in Figure 3-1-A and B which also shows the two different ways of drawing a torsional moment (as a moment with an indicated rotation and as a vector).

Recalling that stresses always are calculated on basis of internal forces and moments, the internal torques will be considered. It is noted, that the internal moment in the shaft shown is constant and is of magnitude T. This is due to, that no matter in which section we cut the shaft, the internal moment in this section must solely maintain equilibrium with the end torque T. We shall in the following see that a shaft subjected to pure torsion is in a state of pure shear.

It is of great importance to note that the theory described in these notes is valid only for solid or hollow cylindrical shafts. All cross-sections which are plane in the unloaded state are assumed to remain plane and undistorted in the loaded state, which is a fair assumption due to axi-symmetry, see Figure 3-3. Furthermore, the angle of twist φ and the shear strain γ must be small (we write $\theta, \varphi << 1$). The material of the shaft must be linear elastic, isotropic (meaning that the material parameters are directionally independent) and homogeneous. Hence, the theory is only valid as long as the stresses are within the elastic range.

Our motivation for learning to master the theory contained in the present chapter is to calculate shear stress and deformation by twist of shafts, which as mechanical components are widely applied in drive train systems. An example of a shaft and a mechanical differential for transmission of torque through a system is shown in Figure 3-2. The present theory is therefore among the most important tools for mechanical design which strength of materials has to offer.

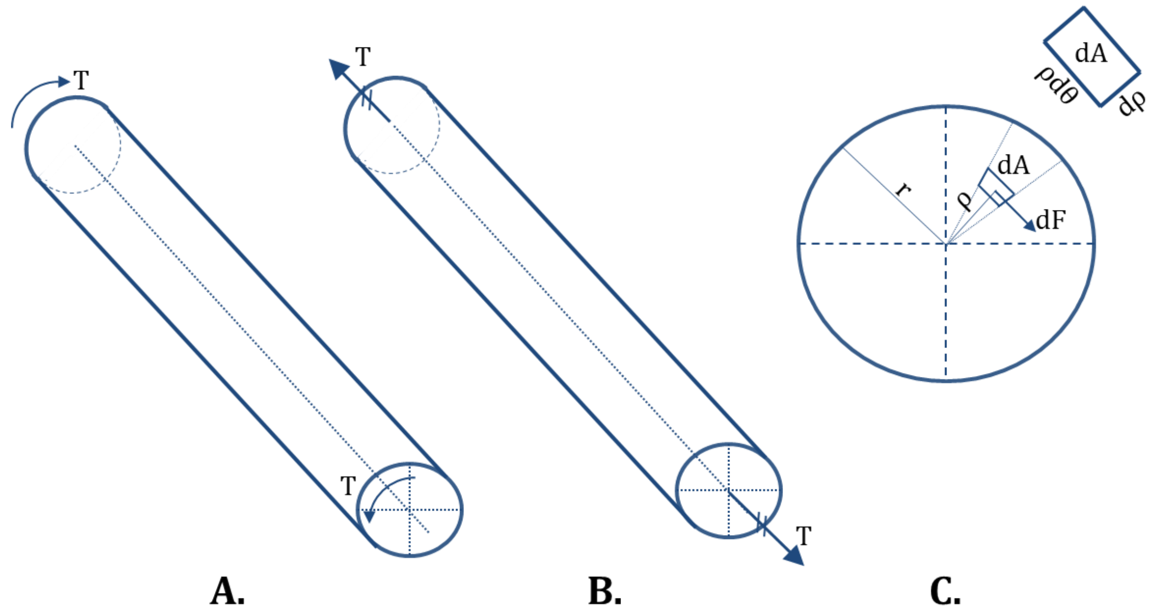

Figure 3-1, A: shaft loaded by end torques shown as rotations, **B**: shaft loaded by end torques shown as vectors, **C** : cross-section of a shaft subjected to torsional loads

Figure 3-2 Shafts and differential for transmission of torque (from Wikimedia, GNU free documentation license)

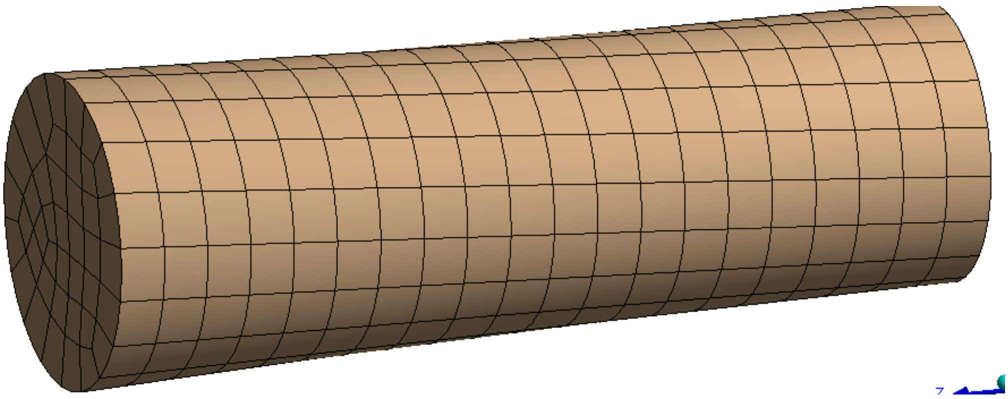

Figure 3-3 Visualization of the state of deformation caused by torsion (obtained by scaling of results obtained by finite element analysis). It is noted that the cross-sections remain plane and undistorted

3.2 Torsional equilibrium

If the cross-section of the shaft in Figure 3-1-A and -B is considered, the internal torque can be calculated by summing up all contributions of force times distance throughout the cross-sectional area. If an infinitesimally small area dA with a radial distance from the centre of the shaft is considered, the force dF contributing to the torque will act perpendicular to the radial direction. The equilibrium of torque T requires that the following condition to be be

$$T = \int \rho \, dF = \int \rho \, \tau \, dA \qquad (3\text{-}1)$$

in which Cauchy's definition of shear stress $\tau = dF/dA$ has been applied in order to rewrite the integral in area dA instead of force dF. However, the distribution of shear stress throughout the cross-section is statically indeterminate, which in this case means that equilibrium of force (or stress) is not sufficient when aiming to find the equilibrium state. It is therefore necessary to consider the deformations in order to determine the stress distribution.

3.3 Mode of deformation: angle of twist and shear strain

In Figure 3-4 a shaft subjected to torsional loads is shown. It is noted that a circular section of radius ρ is considered and not the outer surface defined by the shaft radius, c > ρ. Torsion leads to a twist of the cross-section so the point A is shifted to the position A' in the deformed state. The centre point is denoted O and the angle spanned by the lines OA and OA' is called 'the angle of twist' and is denoted φ. This angle is a measure for how much the shaft has been deformed by torsion. Since the shaft is circular, it can be observed that plane cross-sections remain plane and undistorted. This is not the case for cross-sections with other geometries, which in general may warp in torsion. The line A-B, which is horizontal in the unloaded state together with B-A' spans the angle, which is recognized as the shear strain. The shown element can be observed to be free of normal stress, so a shaft in pure torsion is in a state of pure shear. For small angles, the triangle AA'O can be considered right-angled. The tangents relation therefore gives the following equation

$$tan(\gamma) = \frac{x}{L} \tag{3-2}$$

Equivalently, we get the following expression when considering OAA'

$$tan(\varphi) = \frac{x}{\rho} \tag{3-3}$$

Isolating x in Eq. 3-2 and substituting the obtained expression into Eq. 3-3 yields

$$tan(\gamma) = \frac{tan(\varphi)\rho}{L} \tag{3-4}$$

These equations govern the shaft deformations and are valid for moderately large angles. However, it is difficult to obtain an analytical solution for those due to the harmonic term. We will therefore in the following section simplify those to a form more appropriate for analytical calculations. We note before proceeding that angles always are calculated in radians in structural mechanics.

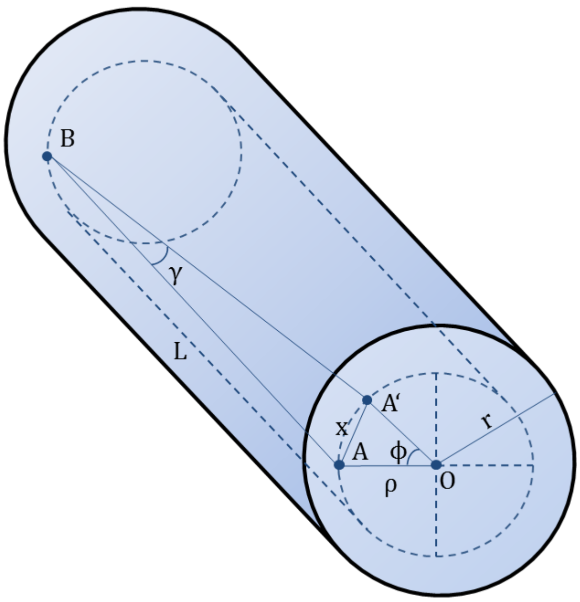

Figure 3-4 Torsional twist and shear strain of a shaft

1.1.1. First order Taylor approximations for small angles

Taylor's formula is a mathematical tool enabling us to approximate a function *f(x)* with a polynomial of *n'th* order on basis of *f(a)* and the n-derivatives in the point *x = a*. Obviously, *f* and it's derivatives to the *n'th - 1* order are required to be continuous differentiable. It is noted that Taylor polynomials not always converge for $n \to \infty$ though this is often the case.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{3-5}$$

The factorial $n!$ Is defined as $n \cdot (n-1) \cdot \ldots \cdot 1$ (example: $4 \cdot 3 \cdot 2 \cdot 1$). The curious reader may have noted, that the factorial $0! = 1$. (per definition) in order for this to make sense. Anyone taking particular interest in this detail, is encouraged to go and bother some mathematician with it. For engineering applications, we in most cases continue by (as fast as possible) simplifying Taylor's formula to a first order approximation

$$f(x) \approx f(a) + f'(a)(x-a)^n \tag{3-6}$$

This is equivalent to approximating *f(x)* by a linear function through the point *f(a)* and *f'(a)*. Applying this to the well-known trigonometric functions sine, cosine and tangents with *a=0* and *x=θ* yields

$$\sin(\theta) \approx \sin(0) + \cos(0)\,\theta = \theta \tag{3-7}$$

$$\cos(\theta) \approx \cos(0) - \sin(0)\theta = 1 \tag{3-8}$$

$$\tan(\theta) \approx \frac{\theta}{1} = \theta \tag{3-9}$$

The trigonometric functions are shown in Figure 3-5 which can also be used to assess the validity range of the conducted approximation. Applying the last result in equation enables us for small angles (*φ, θ << 1*) to perform the following simplification

$$\begin{aligned} tan(\gamma) &= \frac{tan(\varphi)\rho}{L} \\ \to \gamma &\approx \frac{\varphi\rho}{L} \end{aligned} \tag{3-10}$$

OAA' is strictly speaking not right angled, so φ has to be moderately small for the tangent relations to be a fair approximation. However, we could also have obtained this relation on basis of the arclength AA', which would give us the relation $\gamma L = \varphi\rho$, so the result remains valid.
The expression shown above reveals that the shear strain is linearly distributed throughout the cross-section with maximum at the outer surface of the shaft (for *ρ=r*). Hence we have

$$\gamma_{max} \approx \frac{\varphi r}{L} \tag{3-11}$$

The two equations provides the following expression for the shear strain in terms of maximum shear strain

$$\gamma = \gamma_{max}\frac{\rho}{r} \tag{3-12}$$

This enables us to start looking into stress calculations.

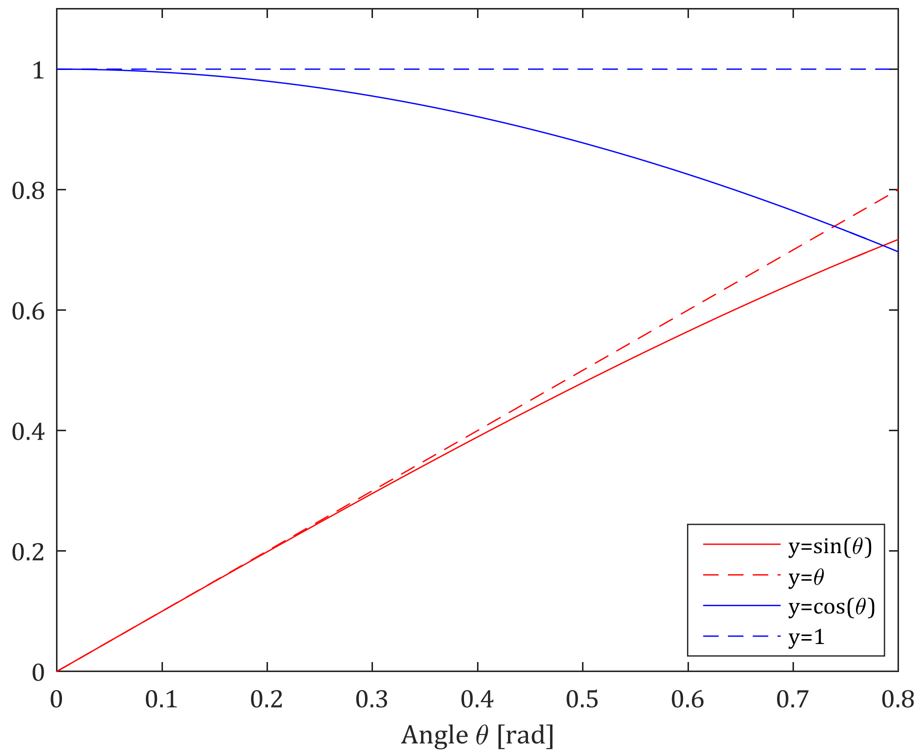

Figure 3-5 First order Taylor approximation of the functions sine and cosine valid for small angles (in radians)

3.4 Calculation of shear stresses due to torsion

Having obtained mathematical expressions for the shaft deformations in terms of the angle of twist φ and the shear strain γ in the previous chapter, our scope is now to use these to derive expressions that can be used for stress calculations. After all, this is what is required for practical design purposes, and is therefore of great importance.

In order to obtain the shear stresses, Hook's law for shear stresses will be applied on the well-known form $\tau = G\gamma$, in which G is the shear modulus and is a material specific parameter. This enables us to relate the shear strain in Eq. 3-12 to shear stresses by multiplying both right and left hand side with G

$$\gamma = \gamma_{max}\frac{\rho}{r}$$
$$\rightarrow G\gamma = G\gamma_{max}\frac{\rho}{r} \quad (3\text{-}13)$$

We immediately recognize these terms as shear stresses and may now write

$$\tau = \tau_{max}\frac{\rho}{r} \quad (3\text{-}14)$$

The physical interpretation of this result is that the shear stresses vary linearly throughout the cross-section as a function of the radial distance to the center of the shaft. Furthermore, it is observed that the shear stresses have maximum value at the outer surface of the shaft (for ρ=*r*).

All that is now left to do, in order to obtain a really awesome equation for design of shafts, is to relate the shear stresses to the internal torques. In order to do so, we will return to considering Figure 3-1, that led us to derive equation 3-1. This can with the knowledge of the state of strain we have gained, be rewritten as

$$T = \int \rho \, \tau \, dA$$

$$= \int \rho \tau_{max} \frac{\rho}{r} \, dA \tag{3-15}$$

$$= \frac{\tau_{max}}{r} \int \rho^2 \, dA$$

We will now for the sake of convenience define the polar moment of inertia in the following fashion

$$J = \int \rho^2 \, dA \tag{3-16}$$

Substituting this into equation 3-15, we obtain the following expressions

$$T = \frac{\tau_{max}}{r} J = \frac{\tau}{\rho} J \tag{3-17}$$

Rearranging this, we obtain what we were after, namely an expression which gets us the torsional stresses

$$\tau(\rho) = \frac{T\rho}{J} \tag{3-18}$$

$$\tau_{max} = \frac{Tr}{J} \tag{3-19}$$

These two formulas can be used for engineering design. The shear stresses can be observed to be linearly distributed over the cross-section with maximum at the outer surface and to be zero in the center of the shaft. This is visualized in Figure 3-6.

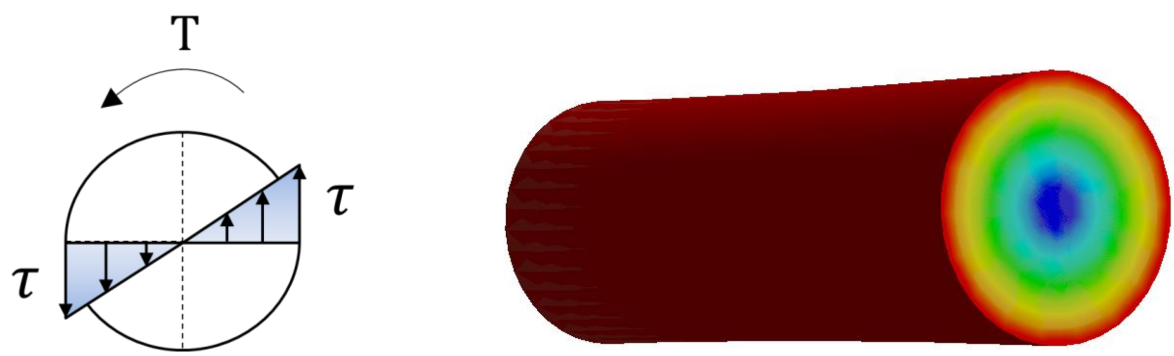

Figure 3-6 Distribution of shear stress due to torsion throughout the cross-section of a shaft

3.5 Calculation of the angle of twist

As a measure of deformations due to torsion, the angle of twist is usually calculated.
We will return to the derived expression for the shear strain given by

$$\gamma = \frac{\varphi\rho}{L} \tag{3-20}$$

Furthermore, we have

$$\gamma = \frac{\tau_{max}}{G} = \frac{Tr}{J}\frac{1}{G} = \frac{Tr}{GJ} \tag{3-21}$$

Letting the two derived expressions for the shear strain equal each other, the following expression for the angle of twist is derived

$$\frac{\varphi r}{L} = \frac{Tr}{JG} \tag{3-22}$$

$$\rightarrow \varphi = \frac{TL}{JG}$$

This expression can be applied for calculation of the angle of twist of a shaft with constant cross-section subjected to end torques as shown in Figure 3-1. For more complex shafts with various diameters and external torques added between the end points, the angle of twist can be calculated by adding the contributions from each separate shaft segments by the equation

$$\varphi = \sum \frac{TL}{JG} \tag{3-23}$$

3.5.1 Calculation of the polar moment of inertia

The polar moment of inertia is now obtained for a solid shaft by evaluation of the integral in the following expression

$$\begin{aligned} J &= \int_A \rho^2 dA = \int_0^{2\pi} \int_0^{r} \rho^2 (\rho d\rho d\theta) \\ &= \int_0^{2\pi} \int_0^{r} \rho^3 d\rho d\theta \end{aligned} \tag{3-24}$$

in which we have utilized that the infinitesimal area considered is given by $dA = \rho d\rho d\theta$. We recall that a double integral is solved by carrying out the first integration in one variable treating the other variable as a constant, before conducting the second integration threating the first variable as a constant. This yields

$$\begin{aligned} \int_0^{2\pi} \int_0^{r} \rho^3 d\rho d\theta &= \int_0^{2\pi} \left[\frac{\rho^4}{4}\right]_0^r d\theta \\ &= \int_0^{2\pi} \frac{r^4}{4} d\theta \\ &= \left[\frac{r^4}{4}\theta\right]_0^{2\pi} = \frac{r^4}{4} 2\pi \end{aligned} \tag{3-25}$$

For a solid shaft we have now obtained the polar moment of inertia. This is a cross-sectional constant given by

$$J_{solid\ shaft} = \frac{\pi}{2} r^4 \tag{3-26}$$

Moments of inertia are additive when calculated around the same axis. Hence, we may obtain the polar moment of inertia for a hollow shaft by subtracting the polar moment of inertia of the inner circle from the polar moment of inertia of the outer circle

$$J_{hollow\ shaft} = \frac{\pi}{2}\left(r_o^4 - r_i^4\right) \tag{3-27}$$

This procedure is illustrated on Figure 3-7.

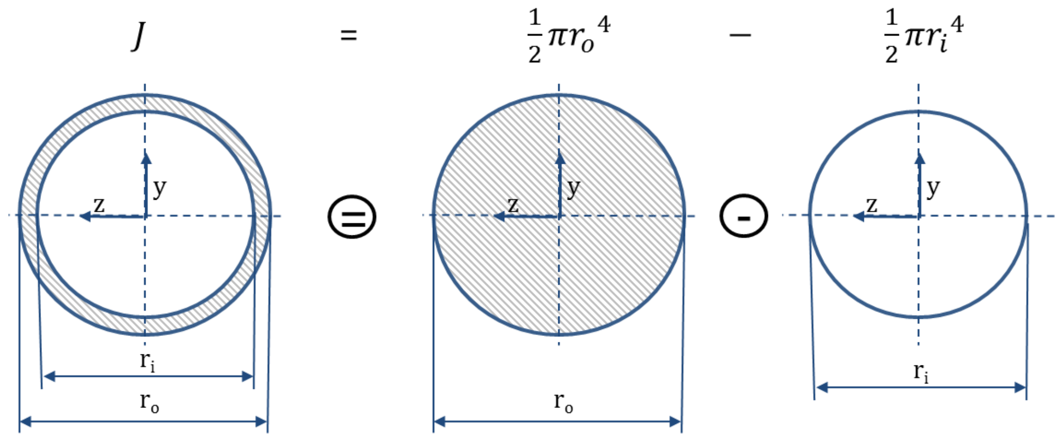

Figure 3-7 Illustration of the process for calculation of the polar moment of inertia for hollow shaft

We will end this chapter with a calculated example demonstrating how shear stresses and angle of twist is calculated for a compound cylinder in torsion.

Calculated example 3A: compound cylinder in torsion

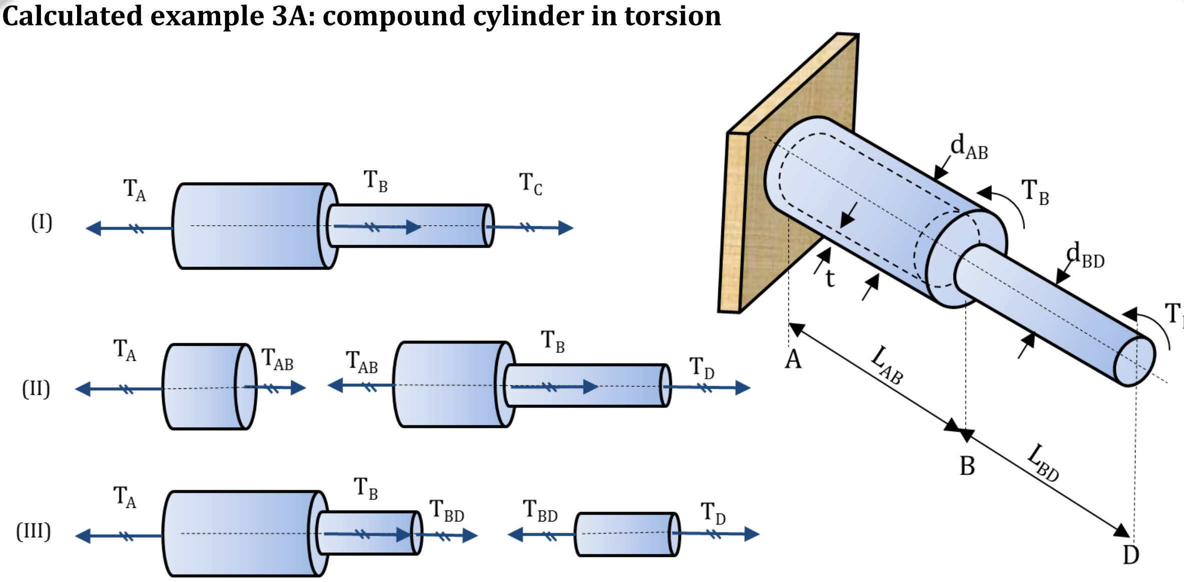

Figure 3-8

The compound cylinder shown in Figure 3-8 has geometry defined by the following set of parameters: *t=10 mm, d_{AB}=125 mm, d_{BD}=80 mm, L_{AB}=300 mm* and *L_{BD}=200mm*. The cylinder is subjected to external torques *T_B=5000 Nm* and *T_D=3000 Nm*. Segment AB is made of steel with shear modulus *G_{AB}=77 GPa* and segment BC is made of aluminum with shear modulus *G_{BD}=25 GPa*.

Calculate **a)** the maximum shear stress in the two segments, **b)** the angle of twist of point D.

Solution

Step 1: The internal torques are determined on basis of the free body diagrams shown in Figure 3-8. For the diagram in Figure 3-8-III, we obtain

$\sum T = 0 \rightarrow -T_{AB} + T_B + T_D = 0 \rightarrow T = T_B + T_D$

In a similar fashion, T_{BD} is obtained on basis of the right segment in Figure 3-8-IV

$\sum T = 0 \rightarrow -T_{BD} + T = 0 \rightarrow T_{BD} = T_D$

[Example 3A continued ...]
Step 2: In order to proceed, the polar moments of inertia for the two segments are required. These are given by

$$J_{AB} = \frac{\pi}{2}\left(r_{AB,o}^4 - r_{AB,i}^4\right) = \frac{\pi}{2}\left(\left(\frac{125\ mm}{2}\right)^4 - \left(\frac{(125-20)mm}{2}\right)^4\right) = 1.20 \cdot 10^7 mm^4$$

$$J_{BD} = \frac{\pi}{2} r_{BD}^4 = \frac{\pi}{2}\left(\frac{80\ mm}{2}\right)^4 = 4.02 \cdot 10^6 mm^4$$

The shear stresses can now be calculated:

$$\tau_{AB} = \frac{T_{AB} r_{AB}}{J_{AB}} = \frac{(5000+3000)\cdot 10^3 Nmm \cdot \frac{125mm}{2}}{1.20\cdot 10^7 mm^4} = 41.7 \frac{N}{mm^2}$$

$$\tau_{BD} = \frac{T_{BD} r_{BD}}{J_{BD}} = \frac{3000\cdot 10^3 Nmm \cdot \frac{80mm}{2}}{4.02\cdot 10^6 mm^4} = 29.8 \frac{N}{mm^2}$$

Step 3: Finally, the angle of twist of point D is obtained as the sum of twists in the two segments

$$\varphi_D = \sum \frac{TL}{GJ} = \frac{T_{AB} L_{AB}}{G_{AB} J_{AB}} + \frac{T_{BD} L_{BD}}{G_{BD} J_{BD}}$$

$$= \frac{(5000+3000)\cdot 10^3 Nmm\ 300mm}{77\cdot 10^3 \frac{N}{mm^2}\cdot 1.20\cdot 10^7 mm^4} + \frac{3000\cdot 10^3 Nmm\ 200mm}{25\cdot 10^3 \frac{N}{mm^2}\cdot 4.02\cdot 10^6 mm^4}$$

$$= 2.59 \cdot 10^{-3}\text{rad} + 5.97 \cdot 10^{-3}\text{rad} = 8.59 \cdot 10^{-3}\text{rad}$$

The twist angle is converted to degrees by

$$8.59 \cdot 10^{-3}\text{rad} \cdot \frac{180}{\pi} \frac{\text{deg}}{\text{rad}} = 0.49\ \text{deg}$$

3.6 Torsion of thin-walled tubes

The theory introduced this far is solely valid when calculating stresses in solid or hollow shafts with circular cross sections. We may extend the theory to apply when calculating stresses in thin-walled tubes of general shape. We will define the shear flow by the equation

$$f = \frac{T}{2A_m} \tag{3-28}$$

which is a measure for how much shear stress that passes through a cross section. The area A_m can in Figure 3-9 be observed to refer to the area spanned by the mid-line of the wall and does not constitute the cross sectional area.

The distribution throughout the wall is however unknown. If a tube is thin-walled, we may assume the shear stress constant throughout the wall thickness. We may then obtain the stresses by dividing through with the thickness t

$$\tau = \frac{f}{t} = \frac{T}{2tA_m} \tag{3-29}$$

As an example of application, we may consider the box shaped tube in Figure 3-10. If this is subjected to a torsional moment, the stresses in the horizontal and vertical faces are given by

$$\tau_{ver} = \frac{T}{2t_1 b'h'} \qquad \tau_{hor} = \frac{T}{2t_2 b'h'} \qquad A_m = b'h' \tag{3-30}$$

It is of the outmost importance to note, that this procedure for stress calculation is only valid for closed tubes with low wall thickness. Open beam profiles like I and T shaped profiles will when subjected to torsion exhibit a behavior including warping, for which cross sections do not remain plane. This cannot be accounted for with the present method.

A procedure for calculation of stresses in solid bars was developed by Timoshenko and is often applied for mechanical design[1]

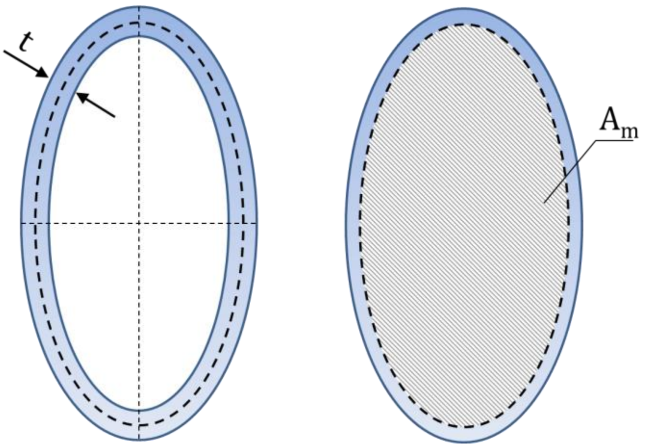

Figure 3-9 Torsion of thin-walled tubes

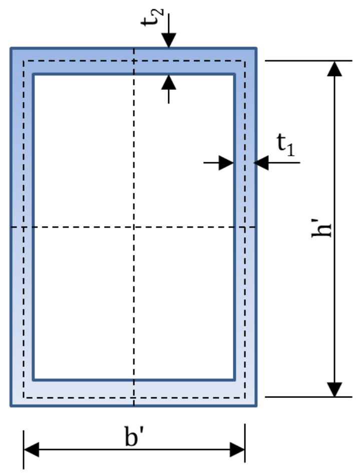

Figure 3-10 Box shaped tube

[1] Online lecture notes from MIT courseware

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Problems

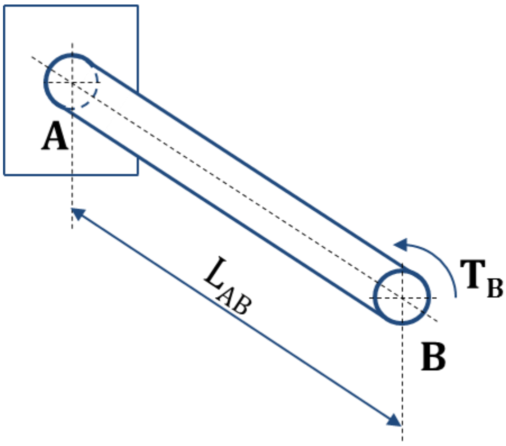


Figure P3.1

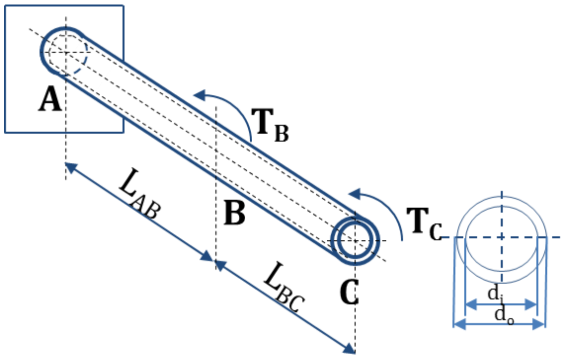


Figure P3.2

Problem 3.1

The shaft shown in Figure P3.1 is of length L_{AB}=1200 mm and radius r_{AB}=20 mm. The shaft is made of titanium with shear modulus G=41.4·10³ N/mm². The design of the shaft must fulfill the following requirements: 1) the shear stress is not to exceed 150 N/mm², 2) the maximum angle of twist is not to exceed 5.0 deg. Calculate the maximum torsional moment that can be applied to the shaft.

Ans: stress based design: T<1.885·10⁶ Nmm, deformation based design: **T<756.7·10³ Nmm**

Problem 3.2

In Figure P3.2, a shaft subjected to the two torsional moments $T_B = T_C = 1000\ Nm$ is shown.

The inner and outer diameter of the shaft is given by d_i=100 mm and d_o=110 mm, and the lengths are given by L_{AB}=1000 mm and L_{BC}=1500 mm. The shaft is made of aluminum with a shear modulus given by G=25·10³ N/mm². Calculate: **a)** the reactional torque moment in the section marked A, **b)** Calculate the stresses in the two segments AB and BC, **c)** Calculate the angle of twist of point C

Ans: a) $T_A = 2000 \cdot 10^3$ *Nmm (directed opposite* T_B *and* T_C*)*

b) τ_{AB}=24.1 N/mm², τ_{BC}=12.1 N/mm²

c) $\phi_C = 0.1756rad + 0.01317rad = 0.03072rad$ (1.76 deg.)

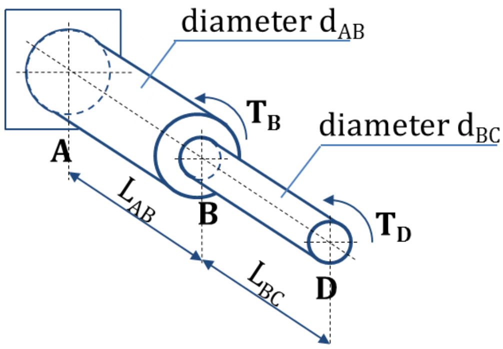


Figure P3.3

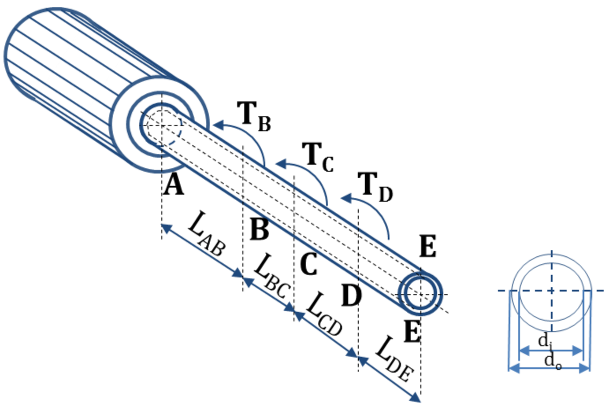


Figure P3.4

Problem P3.3

In Figure P3.3, a compound circular shaft in torsion is shown. The diameters of the shaft are given by d_{AB}=75 mm and d_{BC}=50 mm. The lengths shown in the figure are given by L_{AB}=500 mm and L_{BC}=600 mm. The two torsional moments have directions as specified in figure 3.3 and magnitudes given by $T_B=7500\cdot10^3$Nmm and $T_C=1500\cdot10^3$Nmm. The shaft is made of steel with shear modulus $G=80\cdot10^3 N/mm^2$. **a)** Calculate the stresses in the two segments AB and BC, **b)** Calculate the angle of twist of point C, **c)** Is the angle of twist sufficiently small for the analytical expression to be valid ?

Present a mathematical argument for your answers.

Ans: a) $\tau_{AB}=72.4\ N/mm^2$, $\tau_{BC}=61.1\ N/mm^2$

b) $\phi_C=-0.01207rad+0.01834rad=0.0063rad$ (0.359 deg.)

Problem P3.4

The motor shown in Figure P3.4 rotates the drive shaft AE with constant speed. The drive shaft has constant outer diameter do=120 mm and inner diameter di=80 mm. Furthermore, the drive shaft is subjected to the external torques T_B=3800 Nm, T_C=2600 Nm and T_D=600 Nm. Calculate the stresses in the shaft segments AB, BC, CD and DE.

Ans: $\tau_{AB}=25.71\ N/mm^2$, $\tau_{BC}=11.75\ N/mm^2$, $\tau_{CD}=2.20\ N/mm^2$, $\tau_{DE}=0$

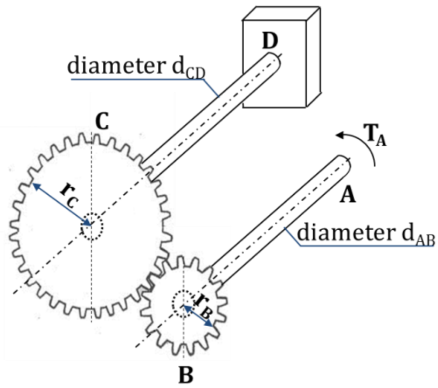


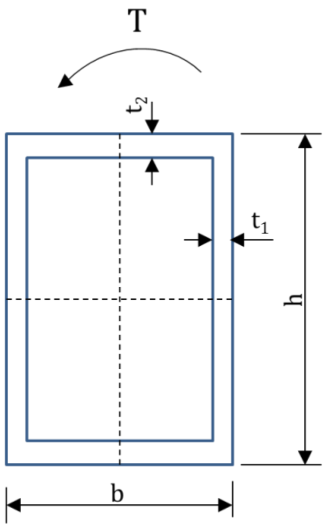
Figure P3.5

Problem 3.5

The two solids shafts shown in Figure P3.5 have geometry given by the parameter set d_{AB}=25 mm, L_{AB}=800 mm, d_{CD}=30 mm and L_{CD}=1200 mm and are connect by gears mounted in B and C with radii r_B=50 mm and r_C=110 mm. The shafts are made of steel with shear modulus $G=77\cdot10^3\ N/mm^2$. If the maximum allowable shear stress is given by $\tau_{allow}=45\ N/mm^2$, **a)** calculate the maximum value of T_A that can be applied, **b)** for this value of T_A, calculate the corresponding angle of rotation of point A.

Ans: **a)** T_{AB}=138.1 Nm, T_{BC}=108.4 Nm

b) ϕ_A=0.132 rad (7.58 deg)



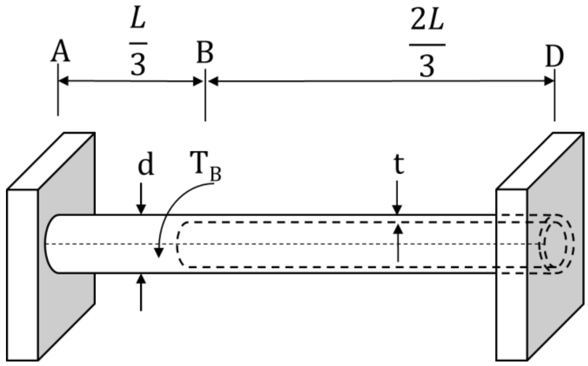


Figure P3.6 **Figure P3.7**

Problem 3.6
A hollow rectangular section is specified by the parameters $h=200\ mm$, $b=120\ mm$, $t_1=10$ mm and $t_2=15\ mm$. The cross-section is subjected to a torsional moment $T=2.2\cdot10^4\ Nm$. Calculate the maximum stress in the cross-section due to torsion.

Ans: $\tau_{max} = 54.1\frac{N}{mm^2}$

Problem 3.7
The pipe shown in figure P3.7 is constrained between two walls preventing it from twisting. The pipe is *2 m* long and with an outer diameter of *110 mm*. Furthermore, section BD is hollow with a constant wall thickness of *20 mm*. The pipe is made of aluminum with a shear modulus of *25 GPa*. If a torque of magnitude $T_B=7.5\cdot10^3\ Nm$, calculate the shear stresses in the sections AB and BD.

Hint: Use the same principle as we applied, when solving statically indeterminate axial load problems to formulate an equation of compability.

Ans: $\tau_{AB} = 20.2\frac{N}{mm^2}$, $\tau_{AB} = 10.1\frac{N}{mm^2}$