# 1. Introduction to strength of materials

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Awesome people in engineering, ch. 1 Augustin-Louis Cauchy (1789–1857) was a French Mathematician, who can be held responsible for proposing solutions to an extremely wide range of problems in mathematics and physics. He graduated from Ecole Polytechnique in 1807 and went into science. During his career he somehow managed to publish more than 700 research papers, which should make it clear to the reader from the very beginning that being geek is in fact a superpower. In the present chapter we will apply Cauchys stress definition However, Cauchy is claimed almost single-handed (and apparently with intent) to have invented complex calculus and made important contributions to the field of wave theory as well.



(Picture from Wikimedia - public domain)

## **1.1** Introduction to the lecture notes

THIS is where you were supposed to start reading ... but I have a strong feeling that I could just skip this section, since no one eventually will end up reading it. But anyway ...

The objective of the present notes is, as the name states, to provide a written introduction to strength of materials. Once upon a time, when I studied engineering myself, it was a prerequisite to buy a book in order to follow a course. When I became a professor, I initially assumed that this was still the case, but soon figured out that times had changed and that I would have to provide some sort of script for my students. I still recommend that the students get a book, but experience shows, that only very few listens to that advice. Therefore, I have attempted to provide a brief written introduction to all mandatory content for a second semester mechanics course for mechanical or mechatronics engineers. The notes are however written late at night (since I have not regained the ability to watch TV after graduating as an engineer). I hope that most errors have been found and corrected in the previous two revisions. If you find mistakes in the notes, please mail me, and I will correct them (at least for future revisions).

However, I figure that many lectures are having the same problem as me ... therefore, if you are a lecturer ...



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If you are a student, remember the first three rules required for learning mechanics:

- 1. Come to the lectures, go to the exercises
- 2. Come to the lectures, go to the exercises
- 3. Mechanics is NOT hard, but if it does not hurt when you start learning something new, you are probably not doing it right



Regards,

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## 1.2 Introduction to strength of materials: stresses and forces

Strength of materials is the discipline related to calculation of stresses and strains in structures and mechanical components. In statics, the concept of static equilibrium was defined and the difference between internal and external forces introduced. However, it is not possible solely on basis of force calculation to assess the structural integrity of a component or structure. This does in layman's terms simply refer to, if the component can sustain the applied loads or will deform beyond the allowable limit, or even break. Strength of materials is obviously a core subject for mechanical and mechatronics engineers, since it enables us to determine by calculation, if the components we design will function as intended or fail. In order to do so, we define the term *stress* as a measure for internal force per area acting inside a structure. Converting our internal forces to stresses by calculations provides us with a measure that contrary to forces can be related to characteristic material values, which we may either measure or look up.

We will furthermore consider the problem related to calculation of deformations in components and structures. On basis of calculated deformations, it will be considered how to calculate *strains* as a measure for how large deformations are relative to the dimensions of the considered component. The intention of the present first chapter is to introduce the general framework, which in the following sets of notes will be applied for analysis of specific mechanical components. The chapter does not contain exercise problems like the following chapters, since we have not started analyzing components yet – we will only consider the general definitions.

## 1.2.1 Recap: the concept of static equilibrium

From statics, we know that the following three relations apply if a plane mechanical system is in static equilibrium

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_z = 0 \tag{1-1}$$

Equation 1-1 basically reads, that we can sum up all forces in the directions of a cleverly chosen coordinate system and furthermore calculate the sum of moments around a point of our own choice. This turned out to be our main tool when calculating reaction forces and moments on structures subjected to external loads. We can extend the plane equilibrium to three dimensions by adding two moment equilibria and an additional force equilibrium

$$\sum M_x = 0 \quad \sum M_y = 0 \quad \sum F_z = 0 \tag{1-2}$$

In the following, we will see that the concept of static equilibrium also serves as basis for calculation of internal forces. If we consider a bar subjected to a single axial load as shown in Figure 1-1-I, equation 1-1 allows us to calculate the reaction force in A. We do so by drawing a *free body diagram*, see Figure 1-1-II, and writing a single equation of equilibrium with a single unknown: the reaction force  $F_A$ . However, we will now imagine that an imaginary cut is introduced between the points A and B separating the bar in two, see Figure 1-1-III. Both sections must remain in static equilibrium, and this can only be maintained if a force acts in the cut we applied. These forces act inside the material and are therefore called *internal* or *inner forces* contrary to the applied loads, which we will refer to as external or *outer forces*. These are the forces we will use for stress calculations.

Before proceeding, we will however need a formal definition of the term *stress*.





Figure 1-1 Example of the internal forces in an idealized console-bar structure



**Figure 1-2 A.** Elastic body subjected to external loads, **B.** imaginary cut through the body with internal force  $\Delta F$  on the surface of the section, **C.** Internal force projected onto the coordinate system of the section [Form slightly simplified]

We will consider an elastic body subjected to external forces  $F_1 \dots F_n$ . Tradition dictates that the body is drawn potato-shaped, but it could have any form. If we add an (imaginary) cut through the body, internal forces must act in the section to maintain equilibrium. We will now for the sake of simplicity define a coordinate system with the x-axis as normal to the section and the y- and z-axes in the plane of the section.



We now let  $\Delta F$  denote the internal force on the area  $\Delta A$  acting inside the material. We may define a stress normal to the section. This is in an average sense given by

$$\sigma_{av} = \frac{\Delta N}{\Delta A} \tag{1-3}$$

For obvious reasons, we call this component *a normal stress*. The two internal forces acting parallel to the section are called shear forces. The corresponding shear stresses in an average sense given by

$$\tau_{y,av} = \frac{\Delta V_y}{\Delta A} \qquad \qquad \tau_{z,av} = \frac{\Delta V_z}{\Delta A}$$
(1-4)

For a point P inside the area  $\Delta A$ , the stress components can now be defined as point values by letting the size of the area turn to zero. Cauchy defined the stresses in the following fashion

$$\sigma_{\chi} = \lim_{\Delta A \to 0} \frac{\Delta N}{\Delta A}$$
(1-5)

$$\tau_y = \lim_{\Delta A \to 0} \frac{\Delta V_y}{\Delta A} \quad \tau_z = \lim_{\Delta A \to 0} \frac{\Delta V_z}{\Delta A}$$
(1-6)

We note that:

- stress calculation is based on internal forces
- stresses are divided into normal and shear components

#### 1.2.3 Inner forces defined in terms of stresses

Now after we have gotten an idea of what a stress is, we notice that stresses often, actually usually, will vary over the considered section. If stresses are known, we can also define the internal forces as three force components and three moment components in terms of stresses. However, since these vary, we will have to consider all infinitesimally small area segments in the section and sum up all force times area terms. This sounds as tedious as it is, and luckily we can use integration to solve this task and integrate a known stress distribution over a cross section to get forces and moments.



Figure 1-3 Internal forces defined as a set of generalized coordinates in terms of stresses



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On basis of Figure 1-1, this gives us the following relations

$$F_{x} = \int \sigma_{x} dA \qquad F_{y} = \int \tau_{xy} dA \qquad F_{z} = \int \tau_{xz} dA$$
$$M_{x} = \int (y \tau_{xz} - z \tau_{xy}) dA \qquad M_{y} = \int z \sigma_{x} dA \qquad M_{z} = \int (-y z \sigma_{y}) dA$$

This might seem a little bit abstract for the time being, but we will get back to this later, since this definition exactly is what is required in order to derive the basic design formulas for stress calculations in shafts and beams.

#### **1.2.4** Principal stresses



Figure 1-4 A solid bar with an oblique section and a axial load

We will consider an axially loaded bar containing an oblique section, see Figure 1-4. The normal and shear stresses in the oblique section are obtained by determining the projection of the force F onto directions normal and parallel to the section. These are divided with the area of the oblique section to obtain the normal and shear stresses. We obtain the following two expressions

$$\sigma = \frac{F\cos\theta}{A} = \frac{F\cos\theta}{\frac{A_0}{\cos\theta}} = \frac{F}{A_0}\cos^2\theta$$
(1-7)

$$\tau = \frac{Fsin\theta}{A} = \frac{Fsin\theta}{\frac{A_0}{cos\theta}} = \frac{F}{A_0}cos\theta\,sin\theta \tag{1-8}$$

The stresses can be observed to be harmonic functions of the angle of the section  $\theta$ . If we plot the stresses for varying values  $\theta$ , we obtain results as shown in Figure 1-5. We note the following:

- The stresses are dependent on the orientation of the section
- A direction for which the shear stresses vanish and leave us with a state of pure normal stress is called *a principal direction*. The corresponding normal stresses are called *principal stresses*. In the present example the first principal direction is the axial direction in which the load is applied and the corresponding principal stress is *F*/*A*<sub>0</sub>. The second principal direction perpendicular to the first direction and the corresponding principal stress is 0. We observed that the shear stress is zero in both principal directions. A transformation to principal coordinates gives us maximum normal stresses and no shear.

We learn more about principal stresses in chapter 7.





Figure 1-5 Normal and shear stress for varying values of  $\theta$ 

### 1.2.5 Saint-Venant's principle

This principle in a qualitative sense deals with the correspondence between stresses and the means by which loads are applied to a given structure. Saint Venant's principle states that one load applied in different fashions will cause local variation in stresses only in a vicinity of the area of action. At a distance from the point or area of action, the stresses will however remain the same, no matter by which means the load was applied. One could also say, that the remaining part of the structure 'does not care' how the load is added. For the bar in Figure 1-6, the stresses in a cross section at a distance from the area where the load is applied will be equal if the following condition is fulfilled:

$$P = \int \sigma_2 dA = \int \sigma_3 dA \tag{1-9}$$

This turns out to extremely useful in many contexts and though it might seem unnecessary abstract, it is eventually worth remembering. Saint-Venant's principle enables us to replace a force acting on a small part of a structure with an equivalent load, that is easier to handle in our calculations, without effecting the stresses at global level.





Figure 1-6 Various ways of applying a load to a structure

## 1.3 Strains and deformations

We have this far solely considered the stresses produced by loads added to a structure and should now be completely on top of the general framework. Therefore, the deformations caused by the loads will now be considered in order to define the term *strain*.

## 1.3.1 Normal strain

A bar subjected to an axial load is again considered, see Figure 1-7. The axial load will not only produce a state of normal stress in the bar, but also cause it to elongate. The elongation appears solely in the axial direction and is denoted  $\delta$ .



**Figure 1-7 Left:** A bar subjected to an axial load, **Right:** failed specimens from a tensile test. We can observe the mode of deformation (figure from <u>Wikimedia</u> – GNU free documentation license)



We now define the normal strain as

$$\varepsilon = \frac{\delta}{L} \tag{1-10}$$

in which L denotes the undeformed length of the bar, which it is fair to apply as long as deformations are small. We write this as  $\delta << L$ . In general, when we do calculations for small deformations, we usually refer to this as application of *small strain theory*. The opposite, calculations for large deflections, relies on *finite strain theory*, and is significantly more complex. We will therefore limit our scope to small deflections. It is furthermore observed, that the strain is unitless.

### 1.3.2 Shear strain

If large shear forces act on a component, these will naturally lead to deformations in the plane of the considered section, see Figure 1-8. While normal forces actually changes the length of faces in a considered section, shear forces will however only distort the shape of the section, see Figure 1-9-B. We therefore simply apply the distortion angle  $\gamma$  as measure for shear strain.



**Figure 1-8 Left:** Assembly in which the load on the bolt is dominated by shear, **Right:** Shear deformation in a bolt after application as shear pin (figure from <u>Wikimedia</u> – CC-licensed)



Figure 1-9 Relation between stress and strain respectively for normal and shear actions



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#### 1.3.3 Hook's law for 1D linear elastic stress

In Figure 1-9 to the right, the deformations caused by normal and shear forces are shown. If stresses seem abstract at this stage, it is probably easier to think about the corresponding strains these cause. The stress and strain terms are namely related. If stresses are linear functions of strains as shown in Figure 1-9 to the left, we say that the material of the considered component obeys Hook's law. The two constants E and G are material parameters called elastic and shear modulus respectively. These are for homogenous isotropic materials (like steel and aluminum) related by the equation  $E = 2G(1 + \nu)$ , in which  $\nu$  is a third material parameters denoted Poisson's ratio. This linear model is a good and reasonable assumption for many metallic materials, if the applied loads do not cause permanent deformations. For other materials, the stress-strain relation would not be linear, but can still be described by a function, which we refer to as a non-linear *material law (or constitutive relation* in general theory of elasticity). The material law serves as our 'grand-link' between stresses and strain and thereby links the applied loads to the deformation of a component. We have now observed that loads might cause either shear or normal stresses and that we may define corresponding strains. We will derive design expressions for this for the most common mechanical components during the course on strength of materials. Examples of such components are shown in Figure 1-10.



**Figure 1-10 A:** Undeformed cylindrical bar with a small area segment *dA*, **B:** Application of axial force will cause elongation leading to a state of pure normal stress and strain uniformly distributed over the cross-section, **C:** Application of a torsional moment will cause twist leading to a state of pure shear stress and strain with a non-uniform distribution, **D:** Adding a moment as external end load will cause bending leading to a state of normal stress and strain with a non-uniform distribution



## 1.4 Material failure

Stress calculation is practical for engineering design, since every material has a limit, where deformations become permanent, in the sense that they do not vanish when a component is unloaded. We say, that plasticity has occurred contrary to elastic material behavior, for which deformations are not permanent. For most design applications, it is undesirable to have plastic deformations, so this limit should not be exceeded (more about this in your course on mechanical elements/technical design). We denote the stress, where the elastic regime ends, the yield stress and denote it  $\sigma_{Y}$ . Once plasticity is encountered, the stress-strain curve will no longer be linear, but further loads may still be applied until the ultimate stress  $\sigma_U$  is reached. When this stress is exceeded, fracture occurs and the material fails. However, once the yield limit is exceeded, two types of material behavior are common and will be considered in the following. Materials may either be ductile, meaning that they possess a certain fracture-toughness in impact tests, and deform significantly in the plastic regime, or brittle, where impact loads easily cause fracture, and very little plastic deformation occurs prior to fracture. As a measure for how brittle a material is, impact tests are often used. The most classic test type is called the Charpy-V (more about this in your course on materials engineering). The stress-strain relation for a typical tensile test of a ductile and a brittle material is shown in Figure 1-11 and Figure 1-12. Glass constitutes an example of a brittle material, since it is highly sensitive to impact loads, and barely deforms plastically before fracture occurs.

We note that the values of the yield stress and fracture stress of metals can be increased by mechanical and thermal processing. However, it is of great importance to remember that the elastic stiffness constants E and G are changed little to none by processing. Therefore, the allowable design stresses for various types of steel and aluminum are different dependent on the material quality. Still, two similar parts manufactured from two different steel types will deform in the same fashion when subjected to equal loads.



Figure 1-11 Ductile material behavior during tensile test





Figure 1-12 Brittle material behavior during tensile test

It is obviously desirable to apply metallic materials with ductile behavior for most applications in mechanical design. In the present course, our scope is limited to the linear elastic range, which is the regime usually needed for mechanical design.

### 1.4.1 Failure criteria

We have now figured out how to design against uni-axial (1-D) stress – we simply have to remain under the yield stress. It was not that hard. The question is now how the handle a multi-axial (2-D or 3-D) state of stress with shear components. This is done using failure criteria. These are basically equations that allow us on basis of a given stress state to calculate a single reference stress, which can be compared with the yield stress. We may transform the stress state to principal coordinates to lose the shear components (that was an important part of the motivation for the development of principal coordinates). The probably widest applied failure criteria is the Von Mises Reference stress criteria, see Figure 1-13. For uni-axial stress this reduces to  $\sigma_{ref} = \sqrt{\sigma^2 + 3\tau^2}$ , but is for a general 3D state of stress given by

$$\sigma_{ref,Von\,Mises} = \sqrt{\frac{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}{2}} = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}}$$
(1-11)



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**Figure 1-13** Visualization of the allowable von Mises reference stress for a 2D state of stress after transformation to principle coordinates (meaning that we have lost the shear components and obtained max. and min. normal stresses  $\sigma_1$  and  $\sigma_2$ )

### Historical example 1A: Brittle failure of the Liberty ships

Ductile steels may turn brittle due to low temperatures or improper heat treatment. The liberty ships were examples of brittle failure in welded sheet metal during WW2 as the welding process was developed. Brittle material behavior is detected by impact tests (denoted fracture toughness). During the failure investigations, it was found that sheet metal in the post-welded state should obtain a fracture toughness of 27 Joule in a Charpy-V impact test to be fit for purpose.





**Figure 1-14** Left: Brittle failure of ship (Picture from <u>Wikimedia</u> – Public domain), Right: Liberty ship Jeremy O'Brien (Foto by author) - maintained by enthusiastic retired naval machinists who love when engineers come by and ask about their ship



#### 1.4.2 Fatigue

We have now figured out, that applied loads cause stresses and once these have been calculated, they can be compared to material values. This comparison enables us to figure out, if plasticity will occur in the analyzed component. However, materials may not only fail due to static overstressing, but also due to material fatigue. This refers to formation of cracks in a component or structure which, if these grow, will lead to fracture and thereby failure of a structure. Fatigue cracks are formed due to cyclic loads and in general, a tensile mean stress leads to a lower component life time (since tensile stresses open cracks). This is commonly accounted for using Smith or Haigh diagrams, but is presently beyond our scope. If a larger number of tensile tests are conducted with cyclic loads of varying amplitudes  $\Delta \sigma$ , it can be determined how many load cycles *N* a test specimen can sustain before failing due to formation of fatigue cracks. Such tests were first conducted by the German railway engineer August Wöhler, who plotted the experimentally obtained results in a double logarithmic coordinate system. This type of diagram is called a S-N or Wöhler-curve, see Figure 1-15. For steel, it can be observed, that a certain limit in number of applied load cycles exists, for which the tensile test specimens have infinite lifetime. This limit is called *the endurance limit*. For aluminum, this limit can be observed not to be present, which is one of the reasons for that steel classically is considered a lot more resistant against fatigue than aluminum. Among other reasons is that aluminums module of elasticity is about one third of steel's, which leads to larger strains that cause the actual fatigue damage.



Figure 1-15 S-N curves for steel and aluminum



## Historical example 1B: Fatigue failure of the Alexander-Kielland drilling rig

The Alexander Kielland semi-submersible drilling rig capsized in 1980 while in operation at the Ekofisk field on the Norwegian Continental shelf killing more than a 100 people. The failure commision concluded that a bracing (shown below) had fractured close a position where electricians had mounted a hydrofon (measuring equipment) by welding close to a man hole. The poorly conducted welding had in combination with the stress concentration from the man hole reduced the fatigue life time of the bracing to an extend, where failure occurred while the rig was in service (though the safety factor against fatigue for offshore structures in the North Sea usually is 10).

The failure report from the Norwegian oil and gas authority (Petroleumsdirektoratet) can be reviewed in your Professors office if he once more forgets to bring it to the lecture.



**Figure 1-16 Left:** The Alexander-Kielland Platform in operation prior to failure (from <u>Wikimedia</u> – free distribution granted), **Right:** sketch of subsea structure with position of failure (from <u>Wikimedia</u> – public domain)

## 1.5 Advanced topic: a brief introduction to the general theory of elasticity

This far we have only considered equilibrium equations formulated in force. However, this is only convenient for components, where a simple system of internal forces can be introduced. For general 3D elasticity this is not the case, see Figure 1-17 and equilibrium equations are formulated directly in stresses.





Figure 1-17 A segment of an elastic body subjected to a three-dimensional state of stress

These are given by the following expressions, in which the *B*-components are the body forces (for example gravity

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + B_x = 0$$
$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + B_y = 0$$
$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + B_z = 0$$

which was derived by expanding the stresses over the considered cube and neglecting higher order terms. The required constitutive equations (material law) is given by

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E}$$
$$\varepsilon_y = -\nu \frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E}$$
$$\varepsilon_z = -\nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

This looks rather horrifying to solve analytically, and that is mostly also the case, which leads us to apply numerical method to obtain a solution for the stresses. However, for students who will specialize in mechanics, these equations will usually re-emerge towards the end of the bachelor or at the beginning of the master program. Now you have seen them and know what you are getting into<sup>1</sup>.

<sup>&</sup>lt;sup>1</sup> A really good online note on introduction to general theory of elasticity from MIT open courseware is available <u>here</u>

